

# Mirrors of curves and their Fukaya categories

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- Plan:
- 1. Mirror construction
  - 2. The fiberwise wrapped Fukaya category
  - 3. Homological mirror symmetry for hypersurfaces
  - 4. Floer theory in trivalent configurations
- } Abouzaid-A.  
A.-Efimov-Katzarkov

$H = f^{-1}(0) \subset (\mathbb{C}^n)^n$  or tonic variety  $V$

hypersurface ( $\dim_{\mathbb{C}} n-1$ ) (actually, defined over Novikov field  $K$ /degenerating family)

$$H = f^{-1}(0), \quad f(x) = \sum_{\alpha \in A \subset \mathbb{Z}^n} c_\alpha t^{p(\alpha)} x^\alpha \quad \text{Laurent polynomial } (t \rightarrow 0)$$

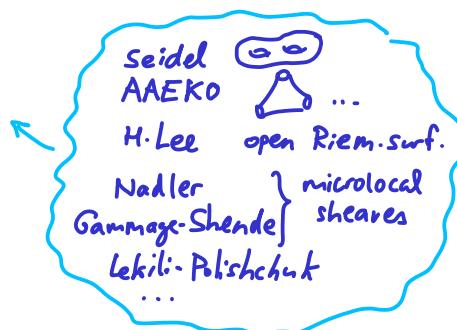
$\Rightarrow$  candidate mirror "Landau-Ginzburg model"  $(Y, W)$

$Y$  = tonic Calabi-Yau  $(n+1)$ -fold,  
(non compact)  $W: Y \xrightarrow{\text{holom.}}$  "superpotential"

expect Fukaya category  $F(H)$   $\longleftrightarrow \sim D_{sg}^b(Y, W)$   
(wrapped)

and coherent sheaves  $D_{coh}^b(H) \longleftrightarrow F(Y, W)$

Abouzaid - A. (see also Cannizzo)  
for abelian var.



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Construction: (Abouzaid-A.-Katzarkov, see earlier work of Hori-Vafa, P. Clarke, ..., also Chan-Lau-Leung, ...)

$$H = f^{-1}(0) \subset (\mathbb{K}^n)^n, \quad f(x) = \sum_{\alpha \in A \subset \mathbb{Z}^n} c_\alpha t^{\rho(\alpha)} x^\alpha \stackrel{t \rightarrow 0}{\sim} \text{Laurent polynomial} \quad (t \rightarrow 0)$$

↑ this means  $x_1^{\alpha_1} \dots x_n^{\alpha_n}$

(or  $H = f^{-1}(0) \subset V$  toric var.,  $f$  section of  $\mathcal{L} \rightarrow V$  with associated polytope  $\simeq \text{Conv Hull}(A)$ ).

→ to find mirror  $(Y, W)$ , let  $\varphi = \text{Trop}(f): \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\varphi(\xi) = \max_{\alpha \in A} (\langle \alpha, \xi \rangle - \rho(\alpha))$ .

Let  $Y$  = toric Kähler (CY) var. with moment polytope  $\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^{n+1} \mid \eta \geq \varphi(\xi)\}$ .

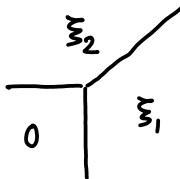
and  $W = w_0 = -z^{(0, \dots, 0, 1)}$  toric monomial vanishing to order 1 on each toric divisor  $\subset Y$ .

(resp  $W = w_0 + w_V$  :  $w_V$  = one monomial for each toric divisor of  $V$ )

Ex. 1:



$$H: \{1 + x_1 + x_2 = 0\}$$



$$\Delta_Y: \eta \geq \max(0, \xi_1, \xi_2)$$

$$Y \simeq \mathbb{C}^3, \quad w_0 = -z_1 z_2 z_3$$

Ex. 1':

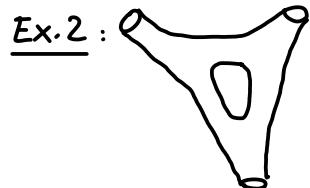


$$\{x_0 + x_1 + x_2 = 0\} \subset \mathbb{P}^2$$

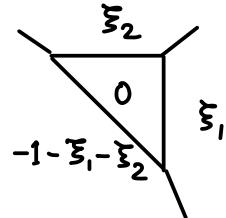


$$(\mathbb{C}^3, \underbrace{-z_1 z_2 z_3}_{w_0} + \underbrace{T(z_1 + z_2 + z_3)}_{w_V})$$

- 3] Remark: • for  $c \neq 0$ ,  $w_0^{-1}(c) \cong (\mathbb{C}^\times)^n$  mirror to ambient  $(\mathbb{C}^\times)^n$ .  
equipped with  $w_V \rightsquigarrow$  mirror to tonic var.  $V$ .
- $w_0^{-1}(0) = \cup$  tonic var's (one for each term in  $f$ ),  
intersecting along  $\text{cut}(w_0) = \cup$   $(n-1)$ -dim. strata.
  - the monodromy of  $w_0$  around origin is "mirror to  $-\otimes \mathcal{O}(H)$ ".



$$H: \left\{ 1 + x_1 + x_2 + \frac{t}{x_1 x_2} = 0 \right\}$$



$$y = \text{Tot } (\mathcal{O}(-3) \rightarrow \mathbb{CP}^2)$$

$\cup, (z_0 : z_1 : z_2)$

$$w_0 = -uz_0 z_1 z_2$$

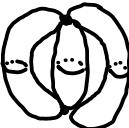


$$\rightsquigarrow (Y, w_0 + w_V = -uz_0 z_1 z_2 + Tu(z_0^3 + z_1^3 + z_2^3))$$

Morse-Bott along smooth cubic elliptic curve  
 $\cong$  usual mirror.

Ex.3:  $H = \text{genus } g \geq 2 \text{ curve} \subset \text{toric surface} \rightarrow (\gamma, \omega_0 + \omega_V)$

$\text{crit}(\omega_0 + \omega_V) = \text{trivalent configuration of } (3g-3) \mathbb{CP}^1\text{'s meeting in } (2g-2) \text{ nodes}$

E.g. genus 2   $\rightarrow \text{crit}(\omega_0 + \omega_V) =$  



5)

## The fiberwise wrapped Fukaya category of $(Y, w_0 + w_V)$ (Abouzaid-A:)

( $\sim$  in proper case, can use Seidel's work; we want to allow fiberwise noncompact Lagrangians)

Remark: A Landau-Ginzburg model  $(Y, w) \xrightarrow{W} \mathbb{C}$  should determine a (non-exact) sector (manifold with boundary),  $Y - \{\operatorname{Re} W < 0\}$ , and its wrapped Fukaya category (stopped at  $\operatorname{Re} W < 0$ ) ??

Instead of stops/sectors, we use monomial admissibility (cf. A. Hanlon's thesis)

Objects: properly embedded Lagrangians  $L \subset Y$  (+ extra data: spin str, grading, ...) which are **tautologically unobstructed** (bound no hol. discs) + **monomially admissible**:

- 1) for  $|w_0| \gg 1$ ,  $\arg(w_0|_L)$  is loc. constant  $\in (-\frac{\pi}{2}, \frac{\pi}{2})$  (ie.  $w_0|_L \in$  union of radial arcs)
- 2) recalling that fibers of  $w_0$  are  $\simeq (\mathbb{C}^*)^n$ , outside of a compact subset of these,   
 $\exists$  finite open cover  $\{U_\nu\}$  and a collection of monomials  $z^\nu$  [the monomials appearing in  $w_V$  if  $V$  is compact, else pick a toric compactif!]   
 st.  $\arg(z^\nu)|_{L \cap U_\nu}$  is loc. constant (e.g.  $\equiv 0$ ).

Note:

- monomial admissibility gives control over discs in Floer products via maximum principle
- use a specific toric Kähler form for which  $\{\arg w_0, \arg z^\nu\}$  Poisson-commute in  $U_\nu$ .

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## Perturbations for $\mathcal{F}(Y, w_0 + w_V)$ :

$L$  admissible  $\leadsto$  flow  $L^t$  (Ham. isotopic to  $L$ ; admissible)

The flow increases the values of  $\arg(w_0)$  and  $\arg(z^\nu)_{|U_\nu}$  at  $\infty$ .

- for  $w_0$  and for  $z^\nu$  which appear in  $w_V$ ,  $\arg \uparrow$  in bounded interval  $\subset (-\pi, \pi)$   
 (the stop at  $\infty$  is  $\{\arg(w_0) = \pi\} \cup \bigcup_{z^\nu \in w_V} \{\arg z^\nu = \pi\} \cap U_\nu$ ) (NO WRAPPING)
- for other monomials  $z^\nu$  not in  $w_0 + w_V$ ,  $\arg \uparrow$  to  $\infty$  (WRAP)

Define

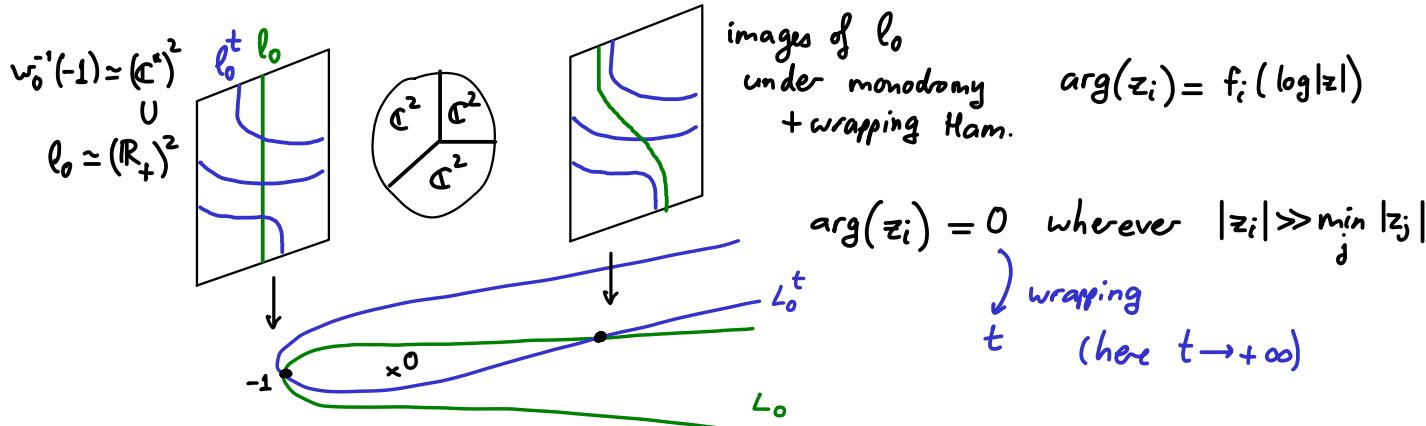
$$\text{hom}(L_0, L_1) := \varinjlim_{t \rightarrow \infty} \text{CF}^*(L_0^t, L_1)$$

under natural continuation maps.

(The fiberwise wrapping Hamiltonians are smoothings of integer piecewise linear functions of the tonic moment map, as in Hanlon)

7  
Example:  $(\mathbb{C}^3, w_0 = -z_1 z_2 z_3)$  (mirror of 

$L_0$  = parallel transport  $\ell_0 = (\mathbb{R}_+)^2 \subset (\mathbb{C}^*)^2 \simeq w_0^{-1}(-1)$  along U-shaped arc.



$$\text{hom}(L_0, L_0) \simeq \text{CW}^*(l_0, l_0) \oplus \text{CW}^*(l_0, l_0)[-1]$$

$\mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}]$        $\hookleftarrow$        $\partial = \text{multiplication by } 1+x_1+x_2$       (Abuzaid-A.)

$$H^* \text{hom}(L_0, L_0) \simeq \frac{\mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}]}{(1+x_1+x_2)} \simeq \text{hom}(\mathcal{O}, \mathcal{O}) \text{ in } D^b \text{Coh}(\text{heart}) \quad \checkmark$$

ring iso.

Similar calculation for  $H \subset (\mathbb{C}^*)^n$  hypersurface:  $H^* \text{hom}(L_0, L_0) \simeq \mathbb{K}[x_i^{\pm 1}] / (f) \simeq \text{End}(\mathcal{O}_H)$   
 $(\partial = \text{mult. by } f = \text{defining eq. of } H)$        $\Rightarrow$  this gives HMS if  $L_0$  generates  $\mathcal{F}(Y, w_0)$ .

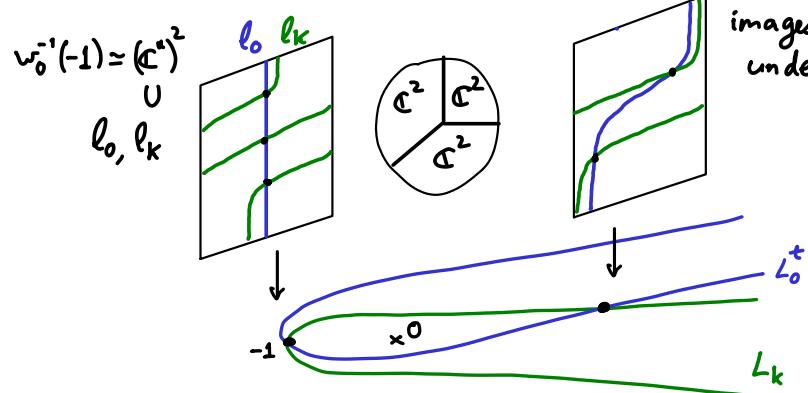
8) Example:  $(\mathbb{C}^3, w_0 + w_V = -z_1 z_2 z_3 + T(z_1 + z_2 + z_3))$  (mirror of  $H: x_0 + x_1 + x_2 = 0 \subset \mathbb{P}^2$ )

Same construction, but don't wrap - only increase  $\arg(z_i)$  slightly.

Fibers  $\{w_0 = -c\}$  are  $\simeq ((\mathbb{C}^*)^2, w_V = T(z_1 + z_2 + \frac{c}{z_1 z_2})) \simeq$  mirror to  $\mathbb{P}^2$ .

Start with  $\ell_k \subset w_0^{-1}(-1) \simeq (\mathbb{C}^*)^2$ ,  $\arg(z_i) = f_i(\log |z|)$  twist by  $2\pi k$  across admissible wrt  $w_V$ , mirror to  $\mathcal{O}_{\mathbb{P}^2}(k)$ .  $HF^*(\ell_0, \ell_k) \simeq H^*(\mathcal{O}_{\mathbb{P}^2}(k))$

$L_k$  = parallel transport  $\ell_k$  along U-shaped arc.



images of  $\ell_0, \ell_k$  under monodromy  $\simeq \ell_0, \ell_{k-1}$  The monodromy of  $w_0$  around origin is mirror to  $-\otimes \mathcal{O}(1)$  [in general,  $-\otimes \mathcal{O}(H)$ ]

$$\Rightarrow \text{hom}(L_0, L_k) \simeq \\ CF^*(\ell_0, \ell_k) \oplus CF^*(\ell_0, \ell_{k-1})[-1] \\ = (x_0 + x_1 + x_2)$$

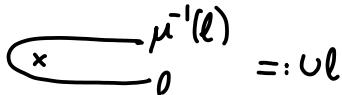
(in general,  $x = \text{def. section of } H$ )

$$\Rightarrow H^0 \text{hom}(L_0, L_k) = H^0(\mathbb{P}^2, \mathcal{O}(k)) / (x_0 + x_1 + x_2) \cong H^0(\mathbb{P}^2, \mathcal{O}(k)). \quad (\text{Abuzeid-A.}) \\ \{x_0 + x_1 + x_2 = 0\} \quad (\text{see also Cannizzaro!})$$

In general: match U-shaped Lagrangians  $\leftrightarrow \mathcal{L}_{IH}$  for  $\mathcal{L} \rightarrow V$  line bundle. ( $\Rightarrow$  HMS mod generation.)

3 The Fukaya categories of  $(Y, \omega_0 + \omega_V)$  and  $(F \simeq (\mathbb{C}^*)^n, \omega_V)$  are related by functors  
 monodromy of  $\omega_0$  around origin

$$\mu \subset F(F, \omega_V) \xrightleftharpoons[\cap]{U} F(Y, \omega_0 + \omega_V)$$

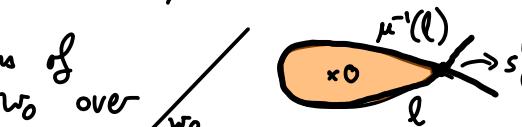
$UL = \text{parallel-transport } l \subset F \text{ (admissible) along U-shape}$    $= UL$   
 $\cap L = \text{ends of } L \subset Y \text{ at } \omega_0 \rightarrow \infty$  (actually  $\in \mathrm{Tw} F(F, \omega_V)$  if  $\omega_0|_L$  has more than one end).

+ exact triangle of functors on  $F(F, \omega_V)$ :  
 (& another in  $F(Y, \omega_0 + \omega_V)$  for the other adjunction)

$$\begin{array}{c} \mu^{-1} \xrightarrow{s} id \\ \downarrow \nu \quad \swarrow \\ \cap U \end{array}$$

(Abouzaid-Ganatra, Seidel).

where  $s = \text{section-counting natural transformation from } \mu^{-1} \text{ to id}$  [Seidel]

$UL \subset F$ ,  $s_\ell^\circ \in CF^0(\mu^{-1}(l), l)$  counts holom-sections of   $\omega_0$  over  $w_0$  over  $l$

In this language, the above calculation is:

$$\mathrm{hom}_Y(UL, UL') \simeq \mathrm{hom}_F(l, \cap UL') \simeq \mathrm{Cone}(\mathrm{hom}_F(l, \mu^{-1}(l')) \xrightarrow{s} \mathrm{hom}_F(l, l'))$$

& homological mirror symmetry is proved by matching this with  $D^b\mathrm{Coh}(H) \xrightarrow[i_*]{i^*} D^b\mathrm{Coh}(V)$

$$\mathrm{hom}_H(i^*\mathcal{L}, i^*\mathcal{L}') \simeq \mathrm{hom}_V(\mathcal{L}, i_*i^*\mathcal{L}') \simeq \mathrm{Cone}(\mathrm{hom}_V(\mathcal{L}, \mathcal{L}' \oplus \mathcal{O}(-H)) \xrightarrow{f} \mathrm{hom}_V(\mathcal{L}, \mathcal{L}'))$$

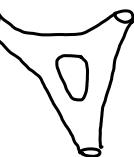
Theorem (Abouzaid-A.)  
"in progress"

$H \hookrightarrow V$  hypersurface in  $(K^n)$  or toric Fano var-  
 $\Rightarrow$  commutative HMS diagram:

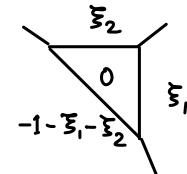
$$\begin{array}{ccc} \mu \subset \mathcal{F}(F, w_V) & \xrightleftharpoons[\cap]{\cup} & \mathcal{F}(Y, w_0 + w_V) \\ \text{HMS for toric vars.} & \simeq \uparrow & \uparrow \text{HMS for hypersurfaces} \\ - \otimes \mathcal{O}(H) \subset D^b \text{Coh}(V) & \xrightleftharpoons[i_*]{i^*} & D^b \text{Coh}(H) \\ & i_x & \end{array}$$

Returning to

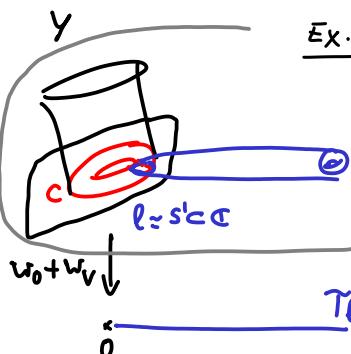
Ex. 2 :



$$H: \left\{ 1 + x_1 + x_2 + \frac{t}{x_1 x_2} = 0 \right\}$$



$$y = \text{Tot } (\mathcal{O}(-3) \rightarrow \mathbb{CP}^2)_{u, (z_0, z_1, z_2)} \\ w_0 = -uz_0 z_1 z_2$$



$\Rightarrow$  Here  $\mathcal{F}(Y, w_0 + w_V) \simeq \mathcal{F}(C)$  via "Thimbles"

Morse-Bott along smooth cubic elliptic curve  $C_T \subset \mathbb{CP}^2$   
 $\cong$  usual mirror.

- for nodal  $C$ ,  
 define  $\mathcal{F}(C)$  as a  
 quotient of  $\mathcal{F}(C_{\text{sm}})$
- This is  $\simeq \mathcal{F}(Y, w)$   
 [M. Jeffs].



For  $H =$



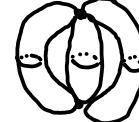
there are similarly mirrors  $C =$



II

For Ex.3:  $H = \text{genus } g \geq 2 \text{ curve} \subset \text{toric surface} \rightsquigarrow (Y, \omega_0 + \omega_V)$

$\text{crit}(\omega_0 + \omega_V) = \text{trivalent configuration of } (3g-3) \mathbb{CP}^1\text{'s meeting in } (2g-2) \text{ nodes}$

genus 2   $\rightsquigarrow \text{crit}(\omega_0 + \omega_V) =$  

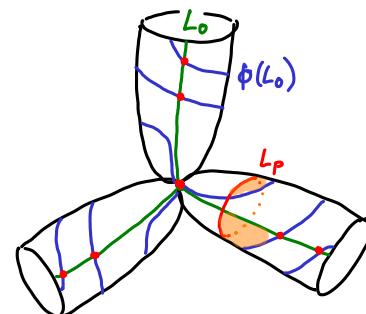
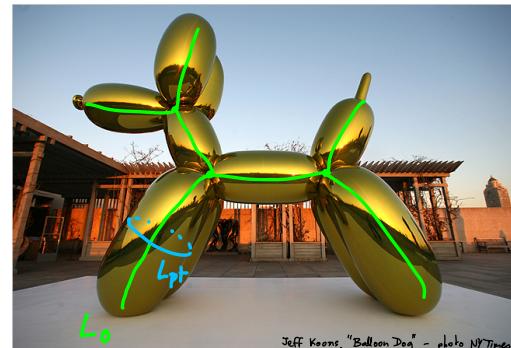
How does one associate a Fukaya category to a trivalent configuration of Riem. surfaces ( $\mathbb{P}^1$ 's &  $\mathbb{C}$ 's)?  
 (A-Efimov-Katzarkov)

Naive construction: line bundles  $\leftrightarrow$  thimbles on trivalent graphs?

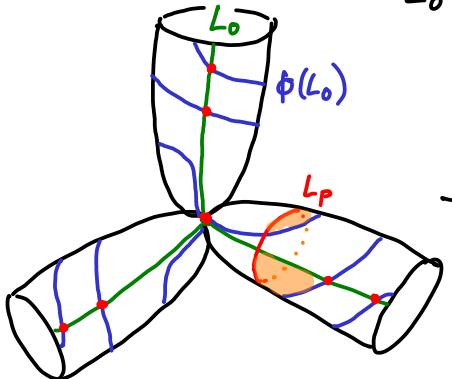
Looking at mirror of  where  $\text{crit}(\omega_0) = \bigcup_3 \mathbb{C}$

would like to consider  $L_0 = \bigcup_3 \mathbb{R}_{\geq 0}$

and its "wrapped Floer homology"



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$L_0 = \bigcup_3 \mathbb{R}_{\geq 0} \subset \bigcup_3 \mathbb{C}$  and its "wrapped Floer homology":

- $CW(L_0, \phi(L_0))$  has 3 infinite series of generators in the cylindrical ends + 1 generator at vertex.

→ should give an additive basis of ring of  $f^{\pm s}$  on  $H = \mathbb{P}^1 - \{0, -1, \infty\}$ ?

- evaluation at points:  $L_p \approx S^1$  in one leg (+ local system)

$\leftrightarrow O_p$ ,  $p = \text{point on } \triangle \text{ near a puncture}$

area enclosed by  $S^1 \leftrightarrow \text{valuation of local coord. at puncture.}$

- \* By analogy with  $W(\square)$  &  $W(\triangle)$ , generators in cyl. ends should be successive powers of inverse of local coord.

Fact: 1 and  $(\text{local coord.})^k$  ( $k \geq 1, i \in \{1, 2, 3\}$ ) are an additive basis of  $O(H)$  (partial fractions!)

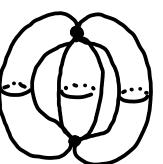
$\hookrightarrow \text{generator at vertex}$        $\hookrightarrow \text{in legs}$

- \* There is no canonical local coord. on  $\mathbb{P}^1 - \{0, -1, \infty\}$  near a puncture.

$\xleftrightarrow{HMS}$  the manner in which (thimble on)  $L_p$  determines an object of  $F(Y, W_0)$  depends on choice of bounding cochain ("framing" in Aganagic-Vafa / Liu/...)

- \* Since a function corresp to a generator in one end is nonzero at points near other punctures, Floer trajectories must propagate through the vertex.

A-model data for  $\Sigma =$



- symplectic form  $\omega$  ( $\leadsto \text{area}(\mathbb{P}_i^*) = A_i > 0$ )  
(+B-field/bulk deformation) ( $\leadsto \text{weight}(\mathbb{P}_i^*) \in \text{val}^{-1}(A_i) \subset K^*$ )
- "chart data" at each vertex  
= choice of  $p_1, p_2, p_3 \in \mathcal{O}(\mathbb{P}^1 - \{0, -1, \infty\})$   
st.  $p_i$  extend to meromorphic functions on  $\mathbb{P}^1$   
with a single pole at the respective puncture.

Preferred choices of  $p_i$ :

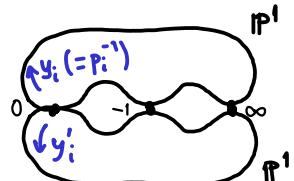
for  $\infty$ :  $\frac{z}{z+1}$  or  $-(z+1)$

for  $0$ :  $-\frac{z+1}{z}$  or  $\frac{1}{z}$

for  $-1$ :  $-\frac{1}{z+1}$  or  $\frac{-z}{z+1}$



Mirror interpretation: near max. degeneration,  = smoothing of

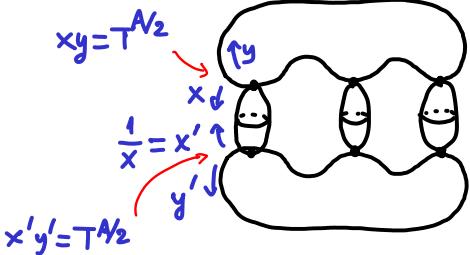


$$y_i y'_i = T^{A_i} \quad \text{smoothing param.}$$

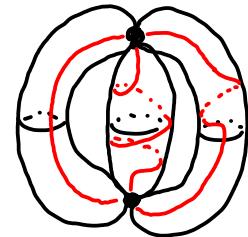
$\uparrow$

or weight of components of  $\Sigma$

or better,

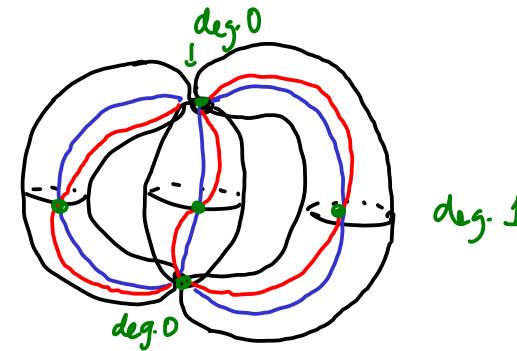
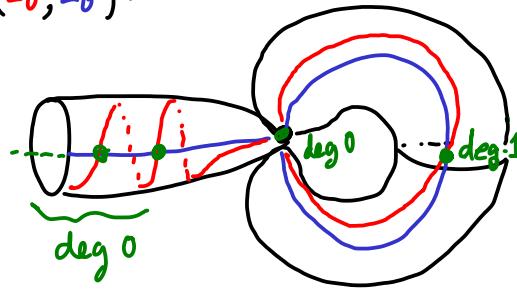


- 14) • Objects of  $F(\Sigma)$  (of line bundle type) = Ex:
- trivalent graphs
- with one arc on each component of  $\Sigma$ ,  
and fixed tangent direction at the vertices  
(+ local systems, trivialized at the vertices)



- To define  $\text{hom}(A, B)$ , we Ham. perturb<sup>n</sup> (w/min. at vertices) to rotate legs of A slightly ccw at vertices + wrap at  $\infty$  in any cylindrical ends of  $\Sigma$ . Each vertex gives a degree 0 generator; other intersections as usual.

Ex:  $\text{hom}(L_0, L_0)$ :



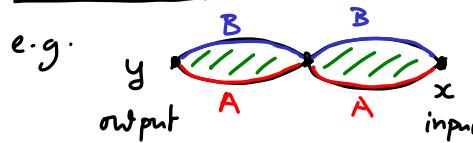
Interpretation: think of a line bundle  $\mathcal{L}$  on mirror curve as built by gluing together

$$\text{[glueing] } \leftrightarrow \mathcal{O} \text{ on } \begin{array}{c} \infty \\ 0 \\ -1 \end{array} \quad \text{and}$$

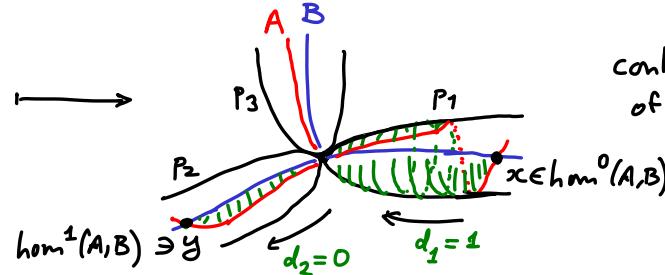
$$\text{[twists] } k \text{ turns } \leftrightarrow \mathcal{O}(k) \text{ on } \begin{array}{c} \mathbb{P}^1 \\ \text{with} \\ \text{fixed trivializations at } 0 \text{ and } \infty. \end{array}$$

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Differential & products on  $\text{hom}(L_i, L_j)$  count rigid (0-dim<sup>!</sup> families) chains/trees of (X<sub>H</sub>-perturbed) holomorphic discs in  $\Sigma$  with boundary on  $L_i \& L_j$ , attached together at vertices of  $\Sigma$ , with weights given by sympl. area (& holonomy of local systems along  $\partial$ ) and multiplicities:



(+ special rule for  $\mu^{\geq 2}$  output at vertex)



contributes to coefft  
of  $y$  in  $\partial x$ .

\* Chart data (loc. coords.  $p_i$  for each leg of each vertex) determine the multiplicities:

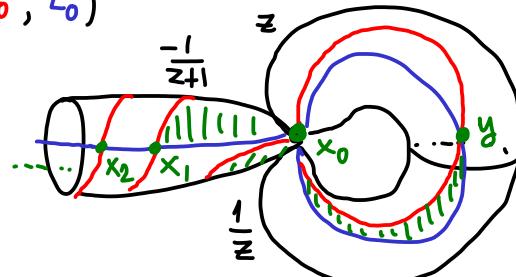
For a vertex where the incoming strip-like end locally covers  $d_1$ -fold ( $d_1 \geq 1$ ) a leg of  $\Sigma$  with local coord.  $p_1$ , and the outgoing one locally covers  $d_2$ -fold ( $d_2 \geq 0$ ) a leg with local coord.  $p_2$ , the local multiplicity contribution := coefficient of  $p_2^{-d_2}$  in the power series expansion of  $p_1^{d_1}$  near the pole of  $p_2$ . Overall multiplicity =  $\prod_{\text{vertices}}$

Eg. in above example, say  $p_1 = \frac{-1}{z+1}$  at  $-1$   
 $d_1 = 1, d_2 = 0$        $p_2 = \frac{1}{z}$  at  $0$        $\rightarrow p_1 = \frac{1}{z+1} = \underbrace{(-1 + (\frac{1}{z})^{-1})}_{\text{multiplicity of } p_2^{-0}} - (\frac{1}{z})^2 + \dots$

(in more complicated examples, typically get binomial coefficients)

Prop: (A.-Efimov-Katzarkov):  $\parallel \mu^k$  satisfy the  $A_\infty$ -relations.

Example:  $\text{hom}(L_0, L_0)$



$\longleftrightarrow \text{Ext}^*(\Theta, \Theta)$  on

$$\partial x_0 = T^E y - T^F y = 0$$

$$\partial x_1 = \pm T^a y$$

Only contribution is

$$\partial x_2 = \pm T^{\text{something}} y \quad (\text{similar})$$

...

$\Rightarrow H^1 \text{hom} = 0$  as expected

$H^0 \text{hom}$  consistent w/  $\Theta(\textcircled{-})$ :

# function with pole order 1!

but  $\exists$  additive basis with one generator  
for each pole order  $\geq 2$ .

$$\left\{ \begin{array}{l} \text{Near } 0: \quad -\frac{1}{z+1} = -1 + \left(\frac{1}{z}\right)^{-1} - \left(\frac{1}{z}\right)^{-2} + \dots \\ \text{Near } \infty: \quad -\frac{1}{z+1} = -z^{-1} + z^{-2} - z^{-3} + \dots \end{array} \right.$$

↑  
no constant term

higher outgoing degree can't reach output  $y$ .

Can in fact get the ring structure on  $H^0 \text{hom}$  to match too!

$$\rightarrow \text{after change of var's, } H^0 \text{hom}(L_0, L_0) \simeq K[x, y] / y^2 = x^3 + ax + b$$

$$x \sim x_2 + cx_1$$

$$y \sim x_3 + \dots$$

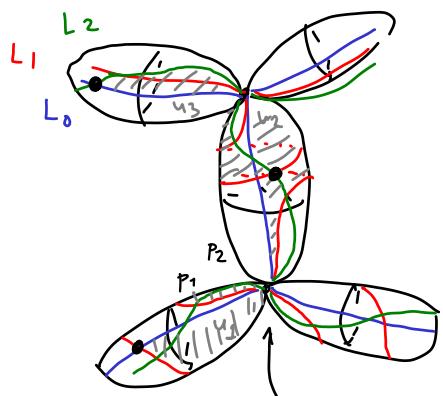
propagation preserves degree of holom. discs,  
so behaves like

but extra branch implements localization wrt  $x(z+1)$

Theorem (A.-Efimov-Katzarkov, in progress)

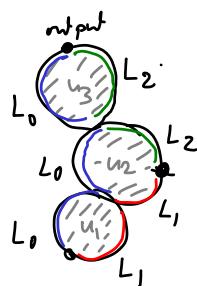
The Fukaya category of a trivalent configuration of  $\mathbb{C}$ 's and  $\mathbb{P}^1$ 's as defined above is derived equivalent to coherent sheaves on the mirror curve constructed by gluing  $(\mathbb{P}^1, 3 \text{ pts})$  and smoothing as prescribed.

Example of  $\mu^2$ . propagating:

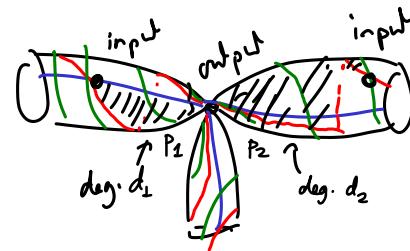


propagation through a node:

$$\text{mult.} = \text{coeff. } P_2^{-d_2} \text{ of } P_2^{d_1} \text{ in expansion of } P_1^{d_1} = \sum_{k \geq 0} c_k P_2^{-k} \text{ at pole of } P_2.$$



Special case: output at a vertex



Expand (partial fractions)

$$P_1^{d_1} P_2^{d_2} = \underbrace{c_0}_{\text{this is the multiplicity.}} + \sum_{k \geq 1} a_k P_1^k + \sum_{l \geq 1} b_l P_2^l$$

