Formal Legendrian and Horizontal embeddings.

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Introduction.

\[ \text{leg}(\mathbb{R}^3) \longrightarrow F\text{leg}(\mathbb{R}^3) \]

\[ T_n(\text{leg}(\mathbb{R}^3)) \xrightarrow{i_*} T_n(F\text{leg}(\mathbb{R}^3)) \]

\[ i_{\ast}[\gamma_0] \neq 0 \]

\[ H_2(\mathbb{R}^4) \xrightarrow{\text{can}} FH_2(\mathbb{R}^4) \]

Casals and del Pino
Formal Legendrian embeddings.

**Definition**

A formal Legendrian embedding in $\mathbb{R}^3$ is a pair $(\gamma, F_s)$ satisfying the following two conditions:

(i) $\gamma : S^1 \to \mathbb{R}^3$ is an embedding.

(ii) $F_s : S^1 \to \gamma^* (T\mathbb{R}^3 \setminus \{0\})$ is a 1-parametric family, $s \in [0, 1]$, such that $F_0 = \gamma'$ and $F_1(t) \in \xi_{\gamma(t)}$. 

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Applications.
Consider the space
\[ \hat{\mathcal{FLeg}}(\mathbb{R}^3) = \{ (\gamma, F_1) \mid \gamma \in \text{Emb}(S^1, \mathbb{R}^3), F_1 \in \text{Maps}(S^1, S^1) \}. \]
We have a natural fibration \( \hat{\mathcal{FLeg}}(\mathbb{R}^3) \to \hat{\mathcal{FLeg}}(\mathbb{R}^3) \).
The fiber over a point \((\gamma, \gamma') \in \hat{\mathcal{FLeg}}(\mathbb{R}^3)\) is
\[ \mathcal{F}(\gamma, \gamma') = \Omega_{\gamma'}(\text{Maps}(S^1, \mathbb{S}^2)). \]
Exact sequence associated to the fibration.

We have the following exact sequence of homotopy groups associated to the fibration:

\[
\cdots \to \pi_2(\text{Emb}(S^1, \mathbb{R}^3)) \to \pi_2(S^2) \oplus \pi_3(S^2) \to \pi_1(\mathcal{L}(\mathbb{R}^3)) \to \pi_1(\text{Emb}(S^1, \mathbb{R}^3)) \oplus \mathbb{Z} \to \pi_0(\mathcal{L}(\mathbb{R}^3)) \to \pi_0(\text{Emb}(S^1, \mathbb{R}^3)) \oplus \mathbb{Z} \to \cdots
\]
Classification Theorem for $\mathcal{F}\text{Leg}(\mathbb{R}^3)$

**Theorem (Folklore)**

*Formal Legendrian embeddings are classified by their topological type as parametrized knots, their rotation number and the Thurston-Bennequin invariant.*

\[ \pi_0(\mathcal{F}\text{Leg}(\mathbb{R}^3)) \cong \pi_0(\mathcal{Emb}(S^1, \mathbb{R}^3)) \oplus \mathbb{Z} \oplus \mathbb{Z} \]
Computing the fundamental group.

**Theorem (Fundamental group of $\mathcal{F}\text{Leg}(\mathbb{R}^3)$. FMP.)**

The sequence

\[
0 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_m \rightarrow \pi_1(\mathcal{F}\text{Leg}(\mathbb{R}^3)) \rightarrow \pi_1(\mathcal{E}\text{mb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z} \rightarrow 0
\]

is exact, where $m \geq 0$.

**Theorem (Fundamental group of $\mathcal{F}\text{Hor}(\mathbb{R}^4)$. FMP.)**

The sequence

\[
0 \rightarrow \mathbb{Z}_2 \rightarrow \pi_1(\mathcal{F}\text{Hor}(\mathbb{R}^4)) \rightarrow \pi_1(\mathcal{E}\text{mb}(\mathbb{S}^1, \mathbb{R}^4)) \oplus \mathbb{Z} \rightarrow 0
\]

is exact.
Geometric interpretation of formal invariants.

\[ \text{rot}_{\pi_1} (\chi^\theta) = \deg (\theta \mapsto \dot{\chi}^\theta (0)) \]
Application: New examples of rigid loops.

Theorem (FMP)

For every knot type $K$ and any Legendrian representative $\tilde{K}$, there exist infinitely many loops of Legendrian embeddings based at $\tilde{K}$ such that:

- they are smoothly trivial.
- they are non trivial as loops of Legendrian embeddings.

\[ \gamma^0(t) = A^0 \cdot \gamma(t) \]

\[ \Theta(\pi) = 5 \in \pi_1(SO(3)) = \mathbb{Z}/2 \]
Geometric interpretation of formal invariants.

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Application: Non-triviality of previous examples in the literature.

T. Kálmán [1] provided infinitely many examples of loops of Legendrians which are smoothly trivial but non-trivial in the space of Legendrians.
Scaling of the supporting knot

(a) Front projection of the knot and the core (in red).

(b) Knot $C^1$—close to the core.

$C^1$—approximation of the knot to the core, seen in the front projection.
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Applications.

Construction of the path of loops into a simplified position.
Thank you very much for your attention!
Bibliography.


