Complexity of (spaces of) Lagrangian Submanifolds.

Octav Cornea

based in part on joint work with
Giovanni Ambrosioni and Paul Biran (ETH)

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I. Symplectic topology background

M^{2n} = smooth manifold.

- (M, ω) is symplectic if ω= 2-form, $d\omega = 0$, ω^n=volume form.
- $L^n \subset (M^{2n}, \omega)$ is Lagrangian if it is a smooth n-dimensional submanifold and $\omega|_L \cong 0$.

Examples

i. $\mathbb{R}^n \subset \mathbb{C}^n$

ii. N= 0-section $\subset T^*N$ = cotangent bundle;

$$(T^*N, d\lambda)\ ,\ \lambda = \text{Liouville form}$$

$\lambda = pdq$ (classical mechanics).

iii. $\mathbb{R}P^n \subset \mathbb{C}P^n$.

iv. Embedded closed curves $\subset \Sigma^2$ = surface.

Good to study Lagrangian submanifolds because:

- Hamiltonian formalism in classical mechanics.
- Applications to dynamics.
- Extension of surface dynamics.
- String theory connections.
- Many, possibly most, interesting properties in symplectic topology can be reformulated in terms of Lagrangian topology.
- Symplectic topology is a friendly cousin of algebraic geometry.
- A descendent of non-linear analysis.
- It draws inspiration and tools from algebraic topology, from homological algebra as well as from PDE's and complex analysis.
- In modern form, it is a new subject (originating in Gromov's *miraculous* paper from 1985) and many fascinating mysteries remain.

Will focus on $M = T^*N$ and $L \subset T^*N$ exact.

$L^n \subset T^*N$ is exact if the inclusion $i_L : L \hookrightarrow D^*N$ satisfies

$$i_L^*(\lambda) = df_L \text{ for some } f_L : L \to \mathbb{R} \text{ (called a primitive of } L) .$$

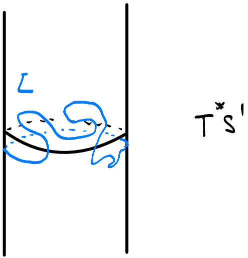
Examples

i. N=0-section, $f_N = 0$.

ii. F= a fiber of $\subset T^*N$, $f_F = 0$.

iii. $g : N \to \mathbb{R}$, $\mathrm{graph}(dg) \subset T^*N$ is an exact Lagrangian.

iv. Other examples that are not graphs:



Fix (N, g) a Riemannian manifold, and the unit disk cotangent bundle D^*N. We will discuss the space:

$$\mathcal{L}ag^{ex}(D^*N) = \{L \subset D^*N \mid L \text{ closed, exact, Lagrangian } \}$$

$\mathcal{L}ag^{ex}(D^*N)$ is the prototype for Lagrangians *nearby* to N in any symplectic manifold.

Weinstein's theorem: if $N \subset (M, \omega)$ is Lagrangian, then $\exists$ neighbourhood of L symplectomorphic to $(D^*_r N, d\lambda)$ (a diffeo $\phi : (M, \omega) \to (M, \omega)$ is a symplectomorphism if $\phi^*\omega = \omega$).

$\Rightarrow \ \forall\ L \subset M$ exact, close enough (in C^0) to $N \ \in \ \mathcal{L}ag^{ex}(D^*N)$.

Aim of talk:

Discuss in what sense $\mathcal{L}ag^{ex}(D^*N)$ is simple and in what sense it is not so. Illustrate some modern symplectic topology in the process.

Symplectomorphisms of (M, ω) form a group $Symp(M, \omega)$.

It contains an important subgroup: Hamiltonian diffeomorphisms

$$Ham(M, \omega) \subset Symp(M, \omega) \; .$$

Given $H : M \times [0, 1] \to \mathbb{R}$, X^H - the Hamiltonian vector field of H - is given by the relation

$$\omega(Y, X^H) = dH(Y), \; \forall \; Y$$

(this gives a one parametric family of v.f. , one for each $t \in [0, 1]$).

A Hamiltonian diffeomorphism is the time-1 diffeo. associated to such a X^H.

L is Hamiltonian isotopic to L',

$$L \sim_H L' \; , \quad \text{if } \exists \; \phi \in Ham(M, \omega), \; L' = \phi(L) \; .$$

II. Conjectures of Arnold and Morse $\rightarrow$ Floer

1. THE Arnold conjecture [middle of '60's] (in simplified form)

$$L, L' \in \mathcal{L}ag^{ex}(D^*N),\ L \sim_H L',\ L \pitchfork L' \ \Rightarrow\ \#(L \cap L') \geq \dim(H_*(L;\mathbb{Z}_2))\ .$$

2. The *nearby* Lagrangian conjecture of Arnold

$$L \in \mathcal{L}ag^{ex}(D^*N)\ \Rightarrow\ L \sim_H N\ .$$

Example

The simplest $L \in \mathcal{L}ag^{ex}(D^*N)$ are graphs, $L = \mathrm{graph}(dg)$, $g : N \rightarrow \mathbb{R}$. For such L: $L \pitchfork N \iff g$ is Morse and $L \cap N = \mathrm{Crit}(g)$. Therefore,

$$\#(N \cap L) \geq \dim(H_*(N;\mathbb{Z}_2))$$

by the Morse inequalities, known since 1920's. Moreover, $L \sim_H N$.

Current status of the two conjectures:

- First conjecture: proved by Floer '87. Broad generalizations established since.
- Nearby Lagrangian conjecture: homotopically true (Fukaya-Seidel-Smith '08, Nadler-Zaslow '08, Abouzaid, Abouzaid-Blumberg 2020's,....) in the sense

$$\forall\ L \in \mathcal{L}ag^{ex}(D^*N),\ L \simeq N$$

(instead of $L \sim_H N$).

The two conjectures $\Rightarrow$ $\mathcal{L}ag^{ex}(D^*N)$ is simple algebraically:

- $\mathcal{L}ag^{ex}(D^*N)/\sim_H = pt$
- each $L \in \mathcal{L}ag^{ex}(D^*N)$ is (algebraically) $\approx$ a graph.

Another point of view - uncovering a more complex structure - has emerged since the end of '90's in relation to Floer's solution to Conjecture 1.

Outline of Floer's approach:

- Let $\mathcal{P}(L, L') = \{\gamma : [0,1] \to M \mid \gamma(0) \in L, \gamma(1) \in L'\}$.
- For $H : M \times [0,1] \to \mathbb{R}$ consider the symplectic action functional: $\mathcal{A}_H : \mathcal{P}(L, L') \to \mathbb{R}$,

$$\mathcal{A}_H(\gamma) = \int_0^1 H(\gamma(t), t)dt - \int_0^1 \lambda(\dot{\gamma}(t))dt \ .$$

- $\mathrm{Crit}(\mathcal{A}_\mathrm{H})$= Hamiltonian orbits of H from L to L'.
- If $H = 0$, $\mathrm{Crit}(\mathcal{A}_\mathrm{H}) = \mathrm{L} \cap \mathrm{L}'$.
- Associate "flow lines to $\nabla \mathcal{A}_H$" even if no flow exists: $u : \mathbb{R} \times [0,1] \to M$, $u(\mathbb{R} \times \{0\}) \subset L$, $u(\mathbb{R} \times \{1\}) \subset L'$ such that:

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla H(u, t) = 0 \text{ (Floer's equation)}$$

J is an almost complex structure compatible with ω.

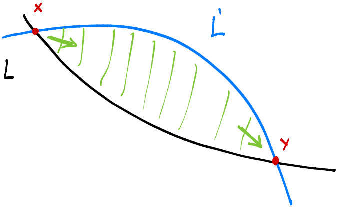


Figure: A Floer strip from x to y for H=0.

- Gromov '85: J-holomorphic curves, as well as perturbed J-curves such as solutions to Floer's equation - called Floer strips - have moduli spaces that are tractable (good compactness and gluing properties).

- Floer '87: use solutions to Floer's equation $\rightarrow$ chain complex: $CF(L, L'; H) = (\mathbb{Z}_2\langle \mathrm{Crit}(\mathcal{A}_{\mathrm{H}})\rangle, \partial)$, $\partial(\mathrm{x}) = \sum \#_2 \mathcal{M}(\mathrm{x}, \mathrm{y}; \mathrm{H})\mathrm{y}$.

- (Floer) $\partial^2 = 0$, $HF(L, L'; H) \cong HF(L; \mathbb{Z}_2)$. In particular, if $H = 0$, $\#(L \cap L') \geq \dim HF_*(L, L') = \dim H_*(L; \mathbb{Z}_2)$.

Floer's complex $CF(L, L'; H)$ is filtered by the values of $\mathcal{A}_H$. As a result:

$$\exists\ d_\gamma \text{ metric on } \mathcal{L}ag^{ex}(D^*N)$$

The metric d_γ - called *spectral metric* - was defined through work of: Viterbo, Schwarz, Oh (1990's and early 2000's),..., Leclercq-Zapolski (2010's), ...

Example

$L = \mathrm{graph}(dg)$, $g : N \to \mathbb{R}$, then

$$d_\gamma(L, N) = \max g - \min g\ .$$

In general, for $\alpha \in H(L; \mathbb{Z}_2)$ put:

$$\mathcal{A}(\alpha) = \inf\{\mathcal{A}(z) \mid z \text{ cycle representing } \alpha \in HF(L, L'; 0) \cong H(L; \mathbb{Z}_2)\}$$

and:

$$d_\gamma(L, L') = \mathcal{A}([L]) - \mathcal{A}([pt])$$

with $[L], [pt] \in H(L; \mathbb{Z}_2)$ the fundamental class and the class of the point, respectively.

Question from now on:

How complicated is the metric space $(\mathcal{L}ag^{ex}(D^*N), d_\gamma)$?

- It is not compact, nor complete.

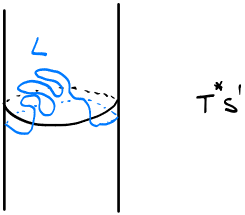


Figure: Curves L can deform to become immersed, and worse.

- Is it bounded ? A conjecture of Viterbo '90's claims yes ! Proved by Shelukhin '23 in some cases.
- We will see that it is not precompact in general (recall: precompact means that completion is compact).
- It is always approximable - a property that extends precompactness and that will be discussed next.

III. Categorical Approximability.

Definition (ABC '25)

Fix $\epsilon > 0$. A metric space (X, d) is *ϵ-approximable* in a triangulated category $\mathcal{C}$ if

i. $X \subset \mathrm{Obj}(\mathcal{C})$ and the metric d extends to a metric d on $\mathrm{Obj}(\mathcal{C})$.

ii. $\exists$ finite $\mathcal{F}_\epsilon \subset \mathrm{Obj}(\mathcal{C})$ such that

$$\forall a \in X, \quad d(a, \mathrm{Obj}\langle \mathcal{F}_\epsilon \rangle^{\Delta}) < \epsilon \ .$$

$\langle \mathcal{F}_\epsilon \rangle^{\Delta}$ is the minimal triangulated sub-category of $\mathcal{C}$ containing $\mathcal{F}_\epsilon$.

X is *approximable* in $\mathcal{C}$ if it is ϵ-approximable for all $\epsilon > 0$.

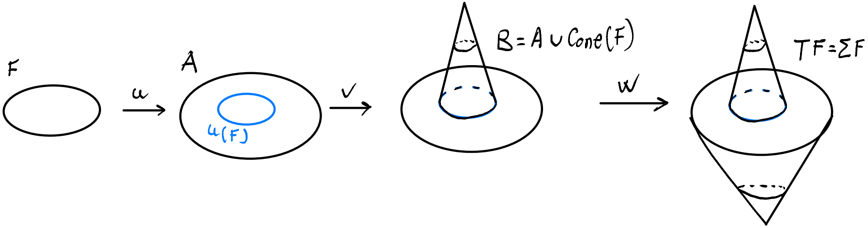
Remark

This is inspired by a notion due to Turing (1938) for groups. Group approximability was further developed since in Geometric Group Theory.

Triangulated categories were introduced in the '60's by Puppe and Verdier (independently). A triangulated category is endowed with a special class of triples (u, v, w) of maps - called *exact triangles*:

$$F \xrightarrow{u} A \xrightarrow{v} B \xrightarrow{w} TF$$

(and a self functor $T : \mathcal{C} \to \mathcal{C}$) with properties mimicking the properties of cone-attachments in topology (with $T \approx$ suspension).



Approximability explicited:

(X, d) is approximable in $\mathcal{C}$ $\iff \forall\ \epsilon > 0\ \ \exists$ finite $\mathcal{F}_\epsilon \subset \mathrm{Obj}(\mathcal{C})$ such that:

$$\forall\ a \in X \qquad \exists \quad \Delta_1, \ldots, \Delta_m$$

sequence of exact triangles in $\mathcal{C}$ (depending on a)

$$\Delta_i \quad : \quad F_i \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow TF_i \ \ ,\ 0 \leq i < m$$

with

$$A_0 = 0\ ,\ \ F_i \in \mathcal{F}_\epsilon\ ,\ \ d(a, A_m) < \epsilon\ .$$

By definition, $\langle \mathcal{F}_\epsilon \rangle^\Delta$ has as objects iterated cones like the A_i's and so:

$$A_m \in \mathrm{Obj}\langle \mathcal{F}_\epsilon \rangle^\Delta \quad , \quad d(a, \mathrm{Obj}\langle \mathcal{F}_\epsilon \rangle^\Delta) < \epsilon\ .$$

Remark

(X, d) is precompact $\Leftrightarrow \forall\ \epsilon > 0, \exists$ finite $\mathcal{F}_\epsilon$ so that

$$\forall a \in X\ ,\ \ d(a, \mathcal{F}_\epsilon) < \epsilon\ .$$

Thus, approximability is a generalization of precompactness.

If (X, d) is approximable it makes sense to define a weighted complexity for each $a \in X$:

Definition

Pick $\epsilon' \geq \epsilon$. The ϵ'-*cone-length* of $a \in X$ (relative to $\mathcal{F}_\epsilon$) is:

$$N(a; \mathcal{F}_\epsilon, \epsilon') = \min\{m \mid \exists \text{ sequence } \Delta_1, \ldots, \Delta_m, \ d(a, A_m) < \epsilon'\} .$$

Key point: (X, d) ϵ- approximable $\Rightarrow$ $N(a; \mathcal{F}_\epsilon, \epsilon') < \infty$, $\forall a \in X$.

Remark

$N(-; -, -)$ is an analogue of the notion of *cone-length* in topology (Ganea '60's; C. '90's). Similar to notions of complexity that appear in homological algebra (Rouquier dimension; generation time; complexity of Dimitrov-Haiden-Kontsevich-Katzarkov). However, in these cases there are no weights.

Here is one use of this complexity.

Lemma

Assume (X, d) *is approximable in* $\mathcal{C}$. *If* (X, d) *is precompact, then* $\exists\ K > 0$ *such that:*

$$\forall\ a \in X, \quad N(a; \mathcal{F}_\epsilon, 2\epsilon) \leq K\ .$$

The proof is immediate: K is the maximal $N(p_i; \mathcal{F}_\epsilon, \epsilon)$ over all the points p_i in an ϵ-net $\{p_1, \ldots, p_k\}$ of X (recall: a precompact metric space admits finite ϵ-nets for each $\epsilon > 0$; ϵ-net $=$ set of centers of ϵ-balls whose union covers X).

To prove (X, d) *is not precompact* show

$$\forall\ \ K > 0, \quad \exists\ a \in X\ , \quad N(a; \mathcal{F}_\epsilon, 2\epsilon) > K\ .$$

Here is one way to show $\exists\ a \in X\ ,\quad N(a; \mathcal{F}_\epsilon, 2\epsilon) > K$

Consider

$$\phi : X \to X$$

and define:

$$h(\phi; a, \mathcal{F}_\epsilon, \epsilon') = \limsup_{n\to\infty} \frac{\log(N(\phi^n a; \mathcal{F}_\epsilon, \epsilon'))}{n}$$

This is a type of *weighted* categorical entropy; a non-weighted variant is due to Dimitrov-Haiden-Kontsevich-Katzarkov, '14.

It remains to show :

$$\exists\ \phi\ , a \text{ with }\ h(\phi : a, \mathcal{F}_\epsilon, 2\epsilon) > 0\ .$$

IV. Triangulated Persistence Categories.

Aim: Produce a class of triangulated categories $\mathcal{C}$ with:

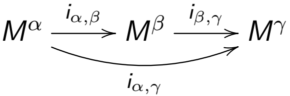
$$(\mathrm{Obj}(\mathcal{C}), d) \text{ metric space .}$$

<u>Recall:</u>

$$M = (\{M^\alpha\}_{\alpha \in \mathbb{R}}, \{i_{\alpha,\beta}\}_{\alpha \leq \beta})$$

is a persistence module if M^α is a module over ring R and

$$i_{\alpha,\beta} : M^\alpha \to M^\beta, \; i_{\beta,\gamma} \circ i_{\alpha,\beta} = i_{\alpha,\gamma}, \; i_{\alpha,\alpha} = id \; .$$



Persistence module machinery has been developed in Topological Data Analysis from '05 on. Some of the roots of the subject are in topology, in Morse theory, in particular.

Definition

A persistence category $\mathcal{C}$ is a category enriched in persistence modules ($\hom_{\mathcal{C}}(a, b) =$ persistence module, the composition respects the persistence structure).

- The objects of a persistence category carry a (pseudo) - metric:

$$d_{\mathrm{int}}(a, b) = \inf\{ r \geq 0 \mid \exists \varphi \in \hom^r(a, b),\ \psi \in \hom^r(b, a),\ \psi \circ \varphi = i_{0,2r}(id_a),\ \varphi \circ \psi = i_{0,2r}(id_b) \}$$

called the interleaving metric.

- The 0-level of a persistence category $\mathcal{C}$ is a category $\mathcal{C}^0$ such that:

$$\mathrm{Obj}(\mathcal{C}^0) = \mathrm{Obj}(\mathcal{C}) \ , \quad \mathsf{hom}_{\mathcal{C}^0}(a, b) = [\mathsf{hom}_{\mathbb{C}}(a, b)]^0$$

Definition (Biran-C.-Zhang '23)

A triangulated persistence category (TPC) is a persistence category $\mathcal{C}$ whose 0-level, $\mathcal{C}^0$, is triangulated (+ some mild additional axioms).

In particular: if $\mathcal{C}$ is a TPC, then

$$(\mathrm{Obj}(\mathcal{C}), d_{\mathrm{int}}) \text{ is a (pseudo) - metric space} \quad .$$

We can talk about approximability in a TPC:

we have a triangulated structure + a canonical metric on the set of objects

V. Back to symplectic topology.

- (N, g) = Riemannian manifold.
- $D^*N \subset T^*N$ unit disk bundle; $(D^*N, d\lambda)$ = symplectic manifold; λ = Liouville form, $\lambda = pdq$.
- Our focus is on $(\mathcal{L}ag^{ex}(D^*N), d_\gamma)$ = set of all closed exact $L \subset D^*N$, d_γ spectral metric.

Theorem (ABC '25 $+\epsilon$)

$(\mathcal{L}ag^{ex}(D^*N), d_\gamma)$ is approximable in a TPC refinement of a derived Fukaya category, $D\mathcal{F}uk(D^*N)$.

If N has < 0 constant sectional curvature, $\exists\ \phi : D^*N \to D^*N$ Hamiltonian diffeomorphism and L such that, for ϵ small enough,

$$h(\phi; L, \mathcal{F}_\epsilon, 2\epsilon) > 0\ .$$

Thus, $(\mathcal{L}ag^{ex}(D^*N), d_\gamma)$ is not precompact (in this case).

Before describing the ingredients in the theorem a little recap.

Summing up:

- $X = (\mathcal{L}ag^{ex}(D^*N), d_\gamma)$ is a prototype for the space of exact Lagrangians near a Lagrangian $N \subset (M, \omega)$, for any M.
- Algebraically, X is very simple. Particularly so if the Arnold nearby conjecture is true in which case $\mathcal{L}ag^{ex}(D^*N)/\sim_H = \{pt\}$.
- As a metric space X is more complicated: it is not complete, and its completion - called sometimes Humillière completion - is not compact (in general).
- However, the metric space X is approximable (a property more general than precompactness) which makes it tractable.
- In particular, there are natural notions of ϵ-weighted complexity, $N(-;-,\epsilon) \in \mathbb{N}$, for the elements of X.
- For a symplectomorphism $\phi : D^*N \to D^*N$, this leads to entropy-like invariants that can be tied to topological entropy (this is related to the negative curvature condition in the statement).

Back to the Theorem:

- $D\mathcal{F}uk(D^*N)$ = TPC refinement of the derived Fukaya category = homological category of filtered A_∞-modules over a filtered A_∞-category, $F\mathcal{F}uk(D^*N)$.
  - This filtered A_∞ category has objects $\mathcal{L}ag^{ex}(D^*N)$.
  - $\hom(L, L') = CF(L, L')$ = Floer complex filtered by symplectic action.
  - μ^k = count of J - holomorphic polygons (these are J-curves with more complicated boundary conditions than Floer strips).

- derived Fukaya category: Gromov '85, Floer '87; Donaldson, Kontsevich, Fukaya - middle of the 90's; Seidel '07.

- Building up on this earlier work, TPC refinement of derived Fukaya category: Biran-C.-Zhang '23, Ambrosioni '24 .

- Variant of the Theorem due to Guillermou-Viterbo-Zhang ('25+ϵ) using micro-local analysis and Tamarkin categories.

Main ingredients to prove the theorem:

- Proof of the Arnold nearby Lagrangian conjecture in homological form due to Fukaya-Seidel-Smith '08 + treatement of Lagrangian cobordisms in Lefschetz fibrations (Biran-C. '16).

- A construction of a Lefschetz fibration structure on T^*N due to Emmanuel Giroux '25.

- A "soft" construction of Morse functions $\varphi : N \to \mathbb{R}$ with large gradient (away from critical points) but small variation.

- A connection between weighted weighted entropy and the bar-code entropy due to Cineli-Ginzburg-Gurel '24 obtained by adjusting some algebraic ideas in DHKK'14.

- A calculation relating bar-code entropy and growth of geodesics.