
ON LOOPS OF LAGRANGIAN TORI

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The *Lagrangian flux* is a cohomology class of L measuring how much a Lagrangian isotopy deviates from a Hamiltonian one, up to homotopy. However, unlike its symplectic analogue, it does not define a group homomorphism on $\pi_1(\mathcal{L}(L))$, the fundamental group of the space of Lagrangians $\mathcal{L}(L)$ isotopic to L . This issue disappears once one keeps track of *monodromy*, that is, the self-equivalence of L induced by the loop:

$$\begin{aligned} \text{FM} : \pi_1(\mathcal{L}(L)) &\longrightarrow H^1(L; \mathbb{R}) \rtimes \text{MCG}(L) \\ \gamma &\longmapsto (\text{Flux}(\gamma), \mathcal{M}(\gamma)) \end{aligned} ,$$

where $\text{MCG}(L) = \pi_0(\text{Diff}(L))$ is the mapping class group of L .

The discreteness of the *flux set* $F(L) := \text{im}(\text{Flux})$ and the *flux-monodromy group* $\Gamma(L) := \text{im}(\text{FM})$ are directly related to topological properties of the Hamiltonian orbit of L , viewed as an unparametrized and a parametrized submanifold, respectively. This correspondence leads to the natural question:

Question : For which Lagrangians L is $\Gamma(L)$ or $F(L)$ discrete?

In the purely symplectic case, the analogous statement is always true: this is the flux conjecture, proved by Ono [Ono06]. In contrast, discreteness can fail in the Lagrangian setting, as our characterization shows:

Theorem 1 ([BCL26]) *Suppose that $\pi_1(M)$ is finite. There exists a Lagrangian $L \subset (M, \omega)$ such that $\Gamma(L)$ is not discrete if and only if the group $\omega(\text{Ker}(c_1 : \pi_2(M) \rightarrow \mathbb{Z}))$ is not discrete.*

Note that nondiscreteness of $\Gamma(L)$ implies nondiscreteness of $F(L)$, but the converse is not generally true. In fact, every symplectic manifold of dimension $2n \geq 6$ has a Lagrangian with nondiscrete flux set [ACLS25].

Despite this general flexibility, we obtain a complete and strikingly rigid picture in one important case.

Theorem 2 ([BCL26]) *Let L be a Lagrangian torus in $\mathbb{C}^2$. Then:*

- (i) $\Gamma(L) = \mathbb{Z} \rtimes \mathbb{Z}_2$ and $F(L)$ is discrete;
- (ii) the flux-monodromy morphism $\text{FM} : \pi_1(\mathcal{L}(L)) \rightarrow \Gamma(L)$ admits a section.

In particular, the Hamiltonian orbit of L is locally path connected and closed (in the C^1 topology between submanifolds), and $\pi_1(\mathcal{L}(L)) = \text{Ker FM} \rtimes (\mathbb{Z} \rtimes \mathbb{Z}_2)$.

This rigidity is special to $\mathbb{C}^2$: In $\mathbb{C}^n$ with $n \geq 3$ [ACLS25] and in many other 4-dimensional manifolds [BK25], the corresponding flux sets are known to be nondiscrete. In fact, we also show that discreteness of $F(L)$ fails for Lagrangian tori in $\mathbb{C}P^2$. The particularity of $\mathbb{C}^2$ relies on the joint availability of almost toric techniques and flexible methods.

Chekanov [Che96] and Yau [Yau12] studied the monodromy of Lagrangian tori in $\mathbb{C}^2$, but their approaches did not allow the construction of a section. But the existence of a section is quite powerful; e.g. it ensures the existence of a subgroup isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2$ in various spaces of Lagrangian tori in symplectic 4-manifolds.

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