What does it mean for two Lagrangians to be near each other?

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Symplectic Topology, Hamiltonian Dynamics, and Persistence Structures:
A conference in honour of Octav Cornea

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Context

Objective: Study the metric geometric of the Hamiltonian orbit $\mathcal{L}\,\mathrm{Ham}(L) := \mathrm{Ham}(M) \cdot L$ of a Lagrangian $L \subseteq M$ in some interesting metric d.

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In this talk, $d = d_{C^1}$ or $d_{\mathrm{Hausdorff}}$.

Outline

1. The C^1 case
   - Some not-quite-so new results
   - Lagrangian flux
2. The Hausdorff case
   - Cieliebak–Mohnke-type capacities
   - Isotopies in cotangent bundles
   - Homologous Lagrangians

Plan

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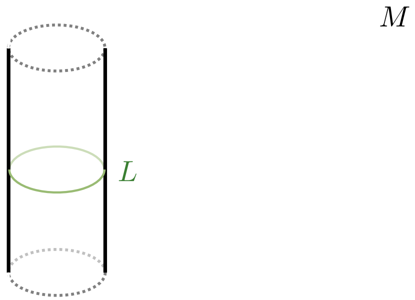
C^1-close Lagrangians: A sketch



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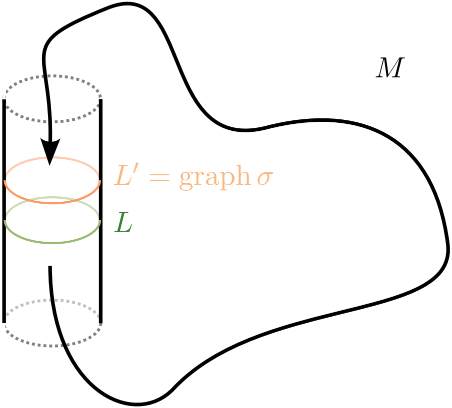

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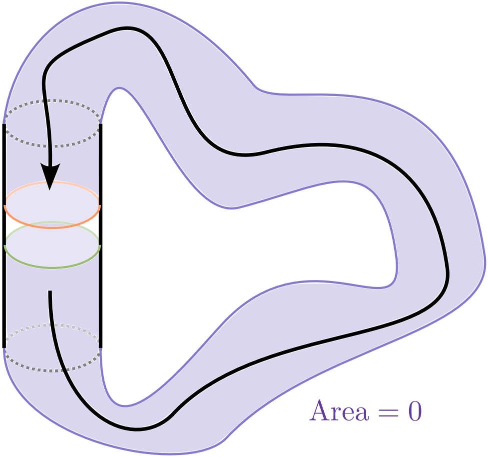

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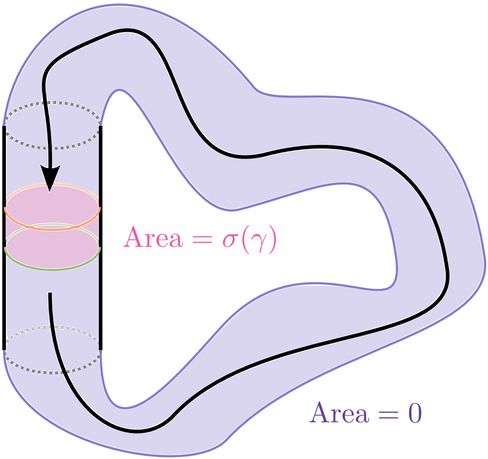

Some positive and negative results

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Theorem (Chekanov '96, Chekanov–Schlenk '16, ACLS '25)

In every symplectic manifold of dimension $2n \geq 6$, there is some L such that $\mathcal{L}\,\mathrm{Ham}(L)$ is *not* C^1-locally path connected.

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Theorem (Kim–Brendel '25)

The same is true in nonmonotone $S^2 \times S^2$'s.

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A Lagrangian loop Λ and a loop γ of L gives a cylinder C_γ^Λ with boundary in L.

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Definition

The *Lagrangian flux* is the map

$$\begin{aligned}\mathrm{Flux}_L : \pi_1(\mathcal{L}, L) &\to H^1(L;\mathbb{R}) = \mathrm{Hom}(\pi_1(L),\mathbb{R}) \\ [\Lambda] &\longmapsto \left([\gamma] \mapsto \mathrm{Area}_\omega(C_\gamma^\Lambda)\right).\end{aligned}$$

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Remark

Contrary to the symplectic case, Flux_L is *not* a morphism and $\mathrm{F}(L)$ is *not* a group, in general.

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Proposition (BCL '26)

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Remark

We can upgrade Flux to a morphism $\mathrm{FM} : \pi_1(\mathcal{L}, L) \to H^1(L;\mathbb{R}) \rtimes \mathrm{MCG}(L)$. There is an analogous statement related discreteness of its image $\Gamma(L)$ to local path connectedness of *parametrized* Lagrangians.

We also get an (almost) complete characterization of when $\Gamma(L)$ can be nondiscrete.

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Theorem (BCL '26)

Let $L \subseteq \mathbb{C}^2$ be a Lagrangian torus. Its flux set $\mathrm{F}(L)$ is discrete, so that $\mathcal{L}\,\mathrm{Ham}(L)$ is C^1-locally path connected (and C^1-closed).

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In either case, the proof relies on computing the group of Lagrangian monodromies in $\mathrm{MCG}(L) = \mathrm{GL}(2;\mathbb{Z})$ using almost toric fibrations.

We then study the action of monodromy on $H^1(L;\mathbb{R})$ by relating it to the action of modular groups on certain hyperbolic surfaces.

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Theorem (ACLS '25)

There is a class of diffeomorphism types $\mathscr{C}$ with the following property. If $L \in \mathscr{C}$ and $\omega(H_2(M, L))$ is discrete, then $\mathcal{L}\,\mathrm{Ham}(L)$ is Hausdorff-locally path connected.

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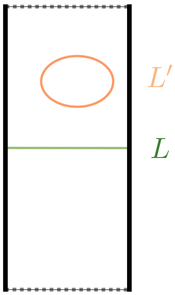
Remark

All examples of L with $\mathcal{L}\,\mathrm{Ham}(L)$ not C^1-locally path connected are also not Hausdorff-locally path connected.

From Hausdorff- to C^1-close: A sketch



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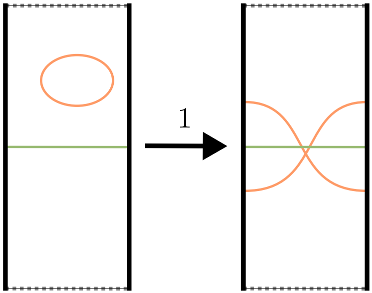

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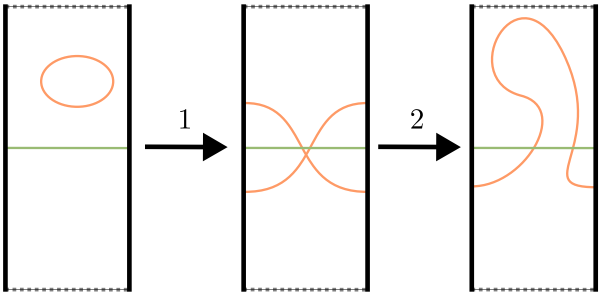

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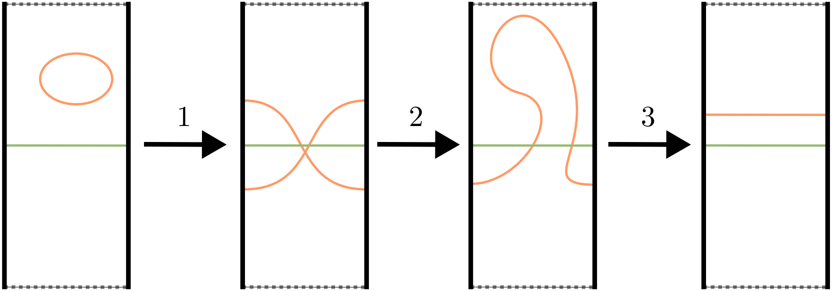

The first reduction

The trick is to show that some capacity of the form

$$c^\star(D^*L) := \inf\left\{\omega(A) > 0 \;\middle|\; A \in H_2(D^*L, L') \neq 0,\ L' \cong L\right\}$$

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How we do it:

- Characterization lemma [ACLS '25]: $\omega(H_2(D^*L, L')) = 0$ implies L' isotopic to exact Lagrangian;
- If $L' \subseteq D^*\mathbb{T}^2$ is not homologous to $\mathbb{T}^2$, it bounds a nonconstant Maslov-2 J-disk [Audin–Lalonde–Polterovich '94].

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- [ACLS '25]: Estimate on Liouville form of nearby Lagrangians and $\omega(H_2(M, L))$ discrete implies exactness;
- Use strong nearby Lagrangian conjecture [Dimitroglou Rizell '19].

Happy birthday Octav!