4. Structure of complex semi-simple Lie algebras

Up to only five exceptions any complex simple Lie algebra is a member of one of three families

$$\mathfrak{sl}(n,\mathbb{C}), \mathfrak{so}(n,\mathbb{C}), \mathfrak{sp}(2n,\mathbb{C}).$$

The last of the three families is associated to *non-degenerate* anti-symmetric complex bilinear forms, in the same way as the second family is associated to non-degenerate *symmetric* complex bilinear forms. Most of what we are going to say in this section is easily verified by inspection for these types and would not need too many abstract arguments.

But we have to go through the abstract results to get at the classification. For the five remaining ones, called F_4, G_2, E_6, E_7 and E_8 there exist explicit constructions, using beautiful mathematics (using Cayley's octonions instead of Hamilton's quaternions). But these constructions do not help very much dealing with for example representation theory. And it is often through representation theory that the applications of exceptional Lie algebra arise. But the approach using Cartan algebra, roots, weights, Weyl group, are sufficiently concrete to still work with these remaining simple complex Lie algebras and their associated simple complex Lie groups. Even the biggest exceptional, E_8, comes up in surprisingly many topics, necessitating work on it.

Let us mention one application. There is a famous and very involved classification of all finite simple groups. Also in that contest there are families, and then some remaining exceptional cases (with the so called Monster group as biggest example). One of these families is the family of alternating groups. To every simple complex Lie algebra, there is an associated family of finite simple groups. For example, $\mathrm{SL}(n,\mathbb{F}_q)$, $\mathrm{SO}(n,\mathbb{F}_q)$, $\mathrm{Sp}(2n,\mathbb{F}_q)$ over the finite fields $\mathbb{F}_q$ (modulo their centers, and variations). There are also twisted versions, like $SU(n,\mathbb{F}(q^2))$. But Chevalley constructed also group versions of the remaining 5 simple complex Lie groups, one for each finite field; and some twisted versions. For this the development of roots, weights, Chevalley bases, etcetera are heavily needed. To construct the simple characters of the simple groups (without its knowledge, the classification of finite simple groups is of limited value), the geometry of the corresponding complex Lie groups, and their homogeneous spaces was needed and very sophisticated algebraic geometry and categorical homological algebra. But our simple complex Lie algebras lie at the center of all this. Any knowledge about these Lie algebras has implications, in the most unexpected contexts (in algebra, geometry and differential equations of course, but also in analysis, physics and number theory).

4.1. We shall now describe some of the general structure of any complex *semi-simple* Lie algebra $\mathfrak{g}$. They depend on the existence of a large abelian Lie subalgebra, called the Cartan algebra.

We shall call an element $X \in \mathfrak{g}$ *semi-simple* if $\mathfrak{g}$ possesses a basis of eigenvectors of $\mathrm{ad}(X)$; we call it semisimple on a representation V if this representation possesses a basis of eigenvectors for the corresponding operator.

Lemma 4.1. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ (where $\mathfrak{g}$ need not be semi-simple), consisting of semi-simple elements only. Then $\mathfrak{h}$ is abelian.*

It follows that there is a basis of $\mathfrak{g}$ that consists of eigenvectors for all $\mathrm{ad}(H), H \in \mathfrak{h}$ simultaneously. Also the normalizer of $\mathfrak{h}$ equals its centralizer.

Proof. Let $X \in \mathfrak{h}$. We want to show that X commutes with any other element of $\mathfrak{h}$. Let $\mu \in \mathbb{C}$ be any eigenvalue for $\mathrm{ad}(X)$ acting on $\mathfrak{h}$, and $Y \in \mathfrak{h}$ a corresponding eigenvector, i.e., $[X, Y] = \mu Y$. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ consisting of eigenvectors for $\mathrm{ad}(Y)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. Then we can express $X = \sum_i c_i e_i$ and $-\mu Y = [Y, X] = \sum_i c_i \lambda_i e_i$ and $0 = [Y, [Y, X]] = \sum_i c_i \lambda_i^2 e_i$. So if the coefficient c_i is non-zero, the eigenvalue λ_i is necessarily 0 and in any case $c_i \lambda_i = 0$. Therefore $-\mu Y = \sum_i c_i \lambda_i e_i = 0$. Since $Y \neq 0$ it follows that $\mu = 0$ and that X acts nilpotently on $\mathfrak{h}$.

Suppose now that $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$ consisting of eigenvectors for $\mathrm{ad}(X)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. We can express any $Y \in \mathfrak{h}$ as $Y = \sum_i c_i e_i$ and $(\mathrm{ad}(X))^m(Y) = \sum_i c_i \lambda_i^m e_i$. Since $\mathrm{ad}(X)$ acts nilpotently there is an m such that $(\mathrm{ad}(X))^m(Y) = 0$. So again, if the coefficient c_i is non-zero, necessarily $\lambda_i = 0$, so $[X, Y] = \sum_i c_i \lambda_i e_i = 0$. So $\mathfrak{h}$ is abelian.

The second statement follows from linear algebra.

Let X be in the normalizer of $\mathfrak{z}$. Let $H \in \mathfrak{h}$, then $[X, H] \in \mathfrak{h}$. Suppose $[X, H] \neq 0$. There is a basis $e_1, \ldots, e_n$ of eigenvectors for $\mathrm{ad}(H)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. We can suppose that $e_1, \ldots, e_r$ is a basis of $\mathfrak{h}$, and hence with 0 eigenvalue since $\mathfrak{h}$ is abelian. We can write $X = \sum_i c_i e_i$ and $[H, X] = \sum_i c_i \lambda_i e_i$. Since $[H, X] \in \mathfrak{h}$, necessarily $c_i \lambda_i = 0$ if $i > r$. On the other hand $\lambda_i = 0$ if $i \leq r$. So we get $[H, X] = 0$. Contradiction. $\square$

Theorem 4.1. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. There exists a non-zero Lie subalgebra $\mathfrak{h}$, called* Cartan subalgebra*, which consists of semisimple elements only and maximal with respect to this property. Then $\mathfrak{h}$ is equal to its own centralizer and equal to its own normalizer. So for $X \in \mathfrak{g}$ we have $X \in \mathfrak{h}$ if and only if X commutes with all the elements of $\mathfrak{h}$ if and only if* $\mathrm{ad}(X)(\mathfrak{h}) \subset \mathfrak{h}$.

We shall not give the proof for the moment.

Example 4.1. Consider $\mathfrak{sl}_n(\mathbb{C})$. Then we can take for $\mathfrak{h}$ the collection of diagonal matrices (with trace 0). Two diagonal matrices always commute, so $\mathfrak{h}$ is indeed an abelian Lie subalgebra. Let $H = \mathrm{diag}(a_1, \ldots, a_n) \in \mathfrak{h}$ be a diagonal matrix with values $a_1, \ldots, a_n$ on the diagonal. Let $E_{ij} \in \mathrm{Mat}(n \times n, \mathbb{C})$ be the matrix with 1 in position ij and 0 elsewhere, $(1 \leq i \leq n, 1 \leq j \leq n)$. A basis for $\mathfrak{h}$ is then $\{E_{ii} - E_{(i+1)(i+1)}; 1 \leq i < n\}$. If we add the E_{ij}, with $i \neq j$, we get a basis for $\mathfrak{sl}_n(\mathbb{C})$.

For $i \neq j$ define the linear function $\alpha_{ij} : \mathfrak{h} \to \mathbb{C}$ (called root) by $\alpha_{ij}(H) := a_i - a_j$. The collection of roots is therefore $\{\alpha_{ij}; i \neq j\} \subset \mathfrak{g}^*$. Then by calculation

$$[H, E_{ij}] = \alpha_{ij}(H) E_{ij},$$

and so E_{ij} is a simultaneous eigenvector for all elements $H \in \mathfrak{h}$ with eigenvalue $\alpha_{ij}(H)$. So indeed any diagonal element is semisimple.

Define $H_{ij} = E_{ii} - E_{jj}$, then

$$[E_{ij}, E_{ji}] = H_{ij}, [H_{ij}, E_{ij}] = 2E_{ij}, [H_{ij}, E_{ji}] = -2E_{ji}$$

and so E_{ij}, E_{ji}, H_{ij} form a basis for a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Let $\pi \in S_n$ be a permutation with permutation matrix L_π. Then conjugation by L_π permutes the entries of the diagonal matrices $H = \mathrm{diag}(a_1, \ldots, a_n)$:

$$\pi \cdot H := L_\pi \, \mathrm{diag}(a_1, \ldots, a_n) L_\pi^{-1} = \mathrm{diag}(a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, \ldots, a_{\pi^{-1}(n)})$$

and permutes the other basis vectors:

$$L_\pi E_{ij} L_\pi^{-1} = E_{\pi(i)\pi(j)}$$

and for any two matrices M, N:

$$L_\pi [M, N] L_\pi^{-1} = [L_\pi M L_\pi^{-1}, L_\pi N L_\pi^{-1}].$$

The symmetric group also acts on $\mathfrak{h}^*$ by $(\pi \cdot \lambda)(H) := \lambda(\pi^{-1} \cdot H)$ and permutes the roots

$$\pi \cdot \alpha_{ij} = \alpha_{\pi(i)\pi(j)}.$$

A basis for $\mathfrak{h}^*$ is formed by the roots (called simple roots)

$$\{\alpha_{i,i+1};\ 1 \leq i < n\}$$

In particular, for $i < j$

$$\alpha_{ij} = \alpha_{i,i+1} + \alpha_{i+1,i+2} + \ldots + \alpha_{j-1,j},$$

and since $\alpha_{ij} = -\alpha_{ji}$ any root is either a sum of simple roots or minus the sum of simple roots.

Let $H = \mathrm{diag}(a_1, \ldots, a_n), H' = \mathrm{diag}(b_1, \ldots, n_n) \in \mathfrak{h}$, then for the Killing form we have

$$B(H, H') = \sum_{i \neq j} \alpha_{ij}(H)\alpha_{ij}(H') = \sum_{i \neq j} (a_i - a_j)(b_i - b_j).$$

So if $B(H, H')$ for all $H' \in \mathfrak{h}$ then $a_1 = a_2 = a_3 = \ldots = a_n$ and the trace of H is 0. Or $H = 0$. So the restriction to $\mathfrak{h} \times \mathfrak{h}$ of the Killing form is non-degenerate.

4.2. Roots and weights. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let V be a $\mathfrak{g}$-representation. For $\lambda \in \mathfrak{h}^*$, define

$$V_\lambda := \{v \in V;\ \forall H \in \mathfrak{h} : H \cdot v = \lambda(H)v\}.$$

If it is non-zero, we call λ a *weight* and V_λ a *weight-space*.

By Theorem 4.1 the $\mathfrak{h}$-representation acting on $\mathfrak{g}$ is the direct sum of its weight spaces

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g};\ \forall H \in \mathfrak{h} : \mathrm{ad}(H)(X) = [H, X] = \lambda(H)X\}.$$

Its non-zero weights are called *roots*, and the corresponding weight spaces are called *root spaces*. For the zero weight we have $\mathfrak{g}_0 = \mathfrak{h}$ (by Theorem 4.1 again, since $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}$). Then for any $\lambda, \mu \in \mathfrak{h}^*$:

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu} \text{ and } \mathfrak{g}_\lambda \cdot V_\mu \subseteq V_{\lambda+\mu}.$$

This is so, since

$$H \cdot (X_\lambda \cdot v_\mu) = [H, X_\lambda] \cdot v_\mu + X_\lambda \cdot H \cdot v_\mu = \lambda(H)(X_\lambda \cdot v_\mu) + X_\lambda \cdot (\mu(H)v_\mu) = (\lambda + \mu)(H)(X_\lambda \cdot v_\mu),$$

for $H \in \mathfrak{h}$, $X_\lambda \in \mathfrak{g}_\lambda$, $v_\mu \in V_\mu$.

Write $\Phi \subset \mathfrak{h}^*$ for the collection of roots, called the *root system*. It has many symmetries, as we will see later. We have a direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

as $\mathfrak{h}$-representation.

Lemma 4.2. *If α is a root (hence by definition non-zero) and $X \in \mathfrak{g}_\alpha$ then* $\mathrm{ad}(X)$ *is nilpotent.*

Proof. Since $\mathrm{ad}(X)(\mathfrak{g}_\beta) \subset \mathfrak{g}_{\alpha+\beta}$, and that we only have finitely many roots, it follows that $\mathrm{ad}(X)$ acts nilpotently. $\square$

4.3. Restriction of the Killing form. Put $(\cdot,\cdot)$ for the restriction to $\mathfrak{h}\times\mathfrak{h}$ of the Killing form $B(\cdot,\cdot)$ (which is not the Killing form of $\mathfrak{h}$). We'll prove that it is non-degenerate. It will follow that for any weight $\lambda \in \mathfrak{h}^*$ there is a unique $\lambda^* \in \mathfrak{h}$ such that for all $H \in \mathfrak{h}$ we have

$$(\lambda^*, H) = \lambda(H).$$

We also get a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ by defining

$$(\lambda,\mu) := (\lambda^*,\mu^*) = \lambda(\mu^*) = \mu(\lambda^*).$$

Proposition 4.1. *(i) Let $\alpha,\beta \in \mathfrak{h}^*$ such that $\alpha+\beta \neq 0$. Then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are perpendicular with respect to the Killing form. It follows that if α is a root, then $-\alpha$ is also a root.*

(ii) The bilinear symmetric form $(\cdot,\cdot)$ on $\mathfrak{h}$ is non-degenerate.

(iii) Let α be any root. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$, then

$$[X,Y] = B(X,Y)\alpha^*.$$

It follows that $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}] = \mathbb{C}\alpha^*$.

(iv) The roots generate $\mathfrak{h}^*$ as vector-space.

(v) Let α be any root. Then $(\alpha,\alpha) = (\alpha^*,\alpha^*) \neq 0$.

Proof. (i) By assumption, there exists an $H \in \mathfrak{h}$ such that $\alpha(H)+\beta(H) \neq 0$. Let $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$, then

$$\alpha(H)B(X_\alpha,X_\beta) = B(\alpha(H)X_\alpha,X_\beta) = B([H,X_\alpha],X_\beta) = -B(X_\alpha,[H,X_\beta]) = -\beta(H)B(X_\alpha,X_\beta).$$

So if $B(X_\alpha,X_\beta) \neq 0$ we must have $\alpha(H)+\beta(H) = 0$. Contradiction.

Since the Killing form is non-degenerate, if $X_\alpha \in \mathfrak{g}_\alpha$ there must be a $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $B(X_\alpha,X_{-\alpha}) = 1$. So if α is a root, necessarily $-\alpha$ is also a root.

(ii) By Cartan's theorem, the Killing form on $\mathfrak{g}$ is non-degenerate. By (ii) every root space is perpendicular to $\mathfrak{g}_0 = \mathfrak{h}$. So if $X \in \mathfrak{h}$ is non-zero, there must be an $Y \in \mathfrak{h}$ such that $B(X,Y) \neq 0$. Or the restriction of the Killing form to $\mathfrak{h}\times\mathfrak{h}$ is non-degenerate.

(iii) Since $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}$ we have $[X,Y] \in \mathfrak{h}$. For any $H \in \mathfrak{h}$

$$(H,[X,Y]) = B(H,[X,Y]) = B([H,X],Y) = \alpha(H)B(X,Y) = (\alpha^*,H)B(X,Y) = (H,B(X,Y)\alpha^*)$$

so $[X,Y] - B(X,Y)\alpha^*$ is perpendicular to all $H \in \mathfrak{h}$. Hence $[X,Y] = B(X,Y)\alpha^*$. By(i), if X is non-zero, then there is an Y such that $B(X,Y) \neq 0$, so $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}] = \mathbb{C}\alpha^*$.

(iv) Otherwise there exists a non-zero $H \in \mathfrak{h}$ such that $\alpha(H) = 0$ for all roots α. So $\mathrm{ad}(H)(\mathfrak{g}_\alpha) = 0$ for every root α. Since $\mathfrak{h}$ is abelian we also have $\mathrm{ad}(H)(\mathfrak{h}) = 0$ and we conclude $\mathrm{ad}(H) = 0$. But since $\mathfrak{g}$ is semisimple the kernel of ad is trivial, hence $H = 0$. Contradiction.

(v) By (iii) there are X,Y as above such that $B(X,Y) = 1$, or $[X,Y] = \alpha^*$. Then $[\alpha^*,X] = \alpha(\alpha^*)X$ and $[\alpha^*,Y] = -\alpha(\alpha^*)Y$. So we get a three dimensional Lie algebra $\mathfrak{k} := \mathbb{C}X + \mathbb{C}Y + \mathbb{C}\alpha^*$.

Let β be any root, then $V = \sum_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$ is a representation for $\mathfrak{k}$, say $\rho : \mathfrak{k} \to \operatorname{End}_{\mathbb{C}} V$. We can calculate the trace of $\rho(\alpha^*)$ as:

$$\operatorname{tr}(\rho(\alpha^*)) = \sum_{i\in\mathbb{Z}} (\dim \mathfrak{g}_{\beta+i\alpha} \cdot (\beta + i\alpha)(\alpha^*)) = \left(\sum_{i\in\mathbb{Z}} i \cdot \dim \mathfrak{g}_{\beta+i\alpha}\right) \cdot \alpha(\alpha^*) + (\dim V) \cdot \beta(\alpha^*).$$

On the other hand,

$$\operatorname{tr}(\rho(\alpha^*)) = \operatorname{tr}(\rho(X)\rho(Y) - \rho(Y)\rho(X)) = \operatorname{tr}(\rho(X)\rho(Y)) - \operatorname{tr}(\rho(Y)\rho(X)) = 0.$$

Supposing that $\alpha(\alpha^*) = 0$, we get $(\dim V) \cdot \beta(\alpha^*) = 0$ so $(\beta, \alpha) = \beta(\alpha^*) = 0$ for every root β. Since the roots generate $\mathfrak{h}^*$, by (iv), it follows that α is in the radical of the form $(,)$. Contradiction, since the form is non-degenerate.

□

4.4. Existence of copies of $\mathfrak{s}l_2(\mathbb{C})$'s. We define the *co-root* $H_\alpha \in \mathfrak{h}$ associated to the root $\alpha \in \Phi$ by normalizing

$$H_\alpha := \frac{2\alpha^*}{(\alpha^*, \alpha^*)},$$

so that $\alpha(H_\alpha) = 2$.

Proposition 4.2. *Let α be a root. There exist $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ such that*

$$[E_\alpha, F_\alpha] = H_\alpha, [H_\alpha, E_\alpha] = 2E_\alpha, [H, F_\alpha] = -2F_\alpha.$$

Hence $E_\alpha, F_\alpha, H_\alpha$ generate a Lie subalgebra isomorphic to $\mathfrak{s}l_2(\mathbb{C})$; we'll denote it by $\mathfrak{s}l_2(\alpha)$.

Proof. By the previous proposition we can choose $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[E_\alpha, F_\alpha] = H_\alpha$. Then $[H_\alpha, E_\alpha] = \alpha(H_\alpha)E_\alpha = 2E_\alpha$, byt the normalization we just made. □

Lemma 4.3. *The co-roots generate $\mathfrak{h}$ as vector space.*

Proof. If this is not the case there exists a non-zero $\lambda \in \mathfrak{h}^*$ such that $\lambda(H_\alpha) = 0$ for all co-roots H_α. So $\lambda(\alpha^*) = (\lambda, \alpha) = 0$ for all roots α, so λ is perpendicular to the space generarate by all roots, i.e. to $\mathfrak{h}^*$, by the previous proposition. So $\lambda = 0$, since $(,)$ is non-degenerate. Contradiction. □

By $\mathfrak{s}l_2(\mathbb{C})$-representation theory the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts on any representation with a basis of eigenvectors with integral eigenvalue. For any $\mathfrak{s}l_2(\alpha)$ the element H_α corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proposition 4.3. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Any finite dimensional $\mathfrak{g}$-representation has a basis of simultaneous eigenvectors for all elements of $\mathfrak{h}$. Or in other words, V is the direct sum of its weight spaces.*

Any weight $\lambda \in \mathfrak{h}^*$ of a $\mathfrak{g}$-representation has the property that $\lambda(H_\alpha) \in \mathbb{Z}$.

Proof. There is a basis of eigenvectors for any H_α, and all H_α's commute in any representation, so we get a simultaneous basis of eigenvectors. Since the H_α's span $\mathfrak{h}$, the same is true for $\mathfrak{h}$. The second statement comes the fact from $\mathfrak{s}l_2$-representation theory just stated. □

Corollary 4.1. *Any root space is one-dimensional. If α is a root, then $n\alpha$ is not a root unless $n = 1$ or $n = -1$.*

So for the Killing form

$$(H, H') = \sum_{\alpha \in \Phi} \alpha(H)\alpha(H').$$

For all roots β_1, β_2 we have that

$$(H_{\beta_1}, H_{\beta_2}) = \sum_{\alpha \in \Phi} \alpha(H_{\beta_1})\alpha(H_{\beta_2}) \in \mathbb{Z}$$

Proof. Consider $\mathfrak{g}$ as a representation for $\mathfrak{sl}_2(\alpha)$. Take V to be the subspace spanned by all weight spaces $\mathfrak{g}_{m\alpha}$ for some m. In particular $\mathfrak{g}_0 = \mathfrak{h}$ is part of it. It is a representation for $\mathfrak{sl}_2(\alpha)$, containing $\mathfrak{sl}_2(\alpha)$ itself. We have that $\mathfrak{g}_{m\alpha}$ consists of eigenvectors for H_α with eigenvalue $m\alpha(H_\alpha) = 2m$.

By $\mathfrak{sl}_2$ representation theory, the number of simple components isomorphic to V_d with d even is the dimension of the 0-eigenspace of H_α (i.e. the dimension of $\mathfrak{h}$) and the number of simple components isomorphic to V_d, with d odd is the dimension of the 1-eigenspace of H_α. We have that $\operatorname{Ker} \alpha$ is a subrepresentation for $\mathfrak{sl}_2(\alpha)$, a direct sum of $(\dim \mathfrak{h} - 1)$ copies of the trivial representation V_0. Since $\mathfrak{sl}_2(\mathbb{C}) \simeq V_2$ we found all simple components isomorphic to V_d, d even. It follows that necessarily all H_α eigenvalues apart from on $\mathfrak{h}$ and $\mathfrak{sl}_2(\alpha)$ itself must be odd. In particular $\mathfrak{g}_{i\alpha} = 0$ if $i = 2, 3, 4, \ldots$. So 2α is never a root. But then $\frac{1}{2}\alpha$ is not a root either. Also necessarily $\dim \mathfrak{g}_\alpha = 1$ also. □

4.5. The Weyl group. We used $\mathfrak{sl}_2(\mathbb{C})$-representation theory to prove that any root space is one dimensional and that if α is a root, then $2\alpha, 3\alpha, \ldots$ is not a root. We have some more applications.

For any root α define the linear map $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

Then $s_\alpha^2(\lambda) = \lambda$, and so s_α is a linear involution. We call s_α a *reflection* since it leaves the space perpendicular to α fixed, and maps α to $-\alpha$.

Proposition 4.4. *Fix a root α. Let V be a $\mathfrak{g}$-representation with some weight λ. Then $s_\alpha(\lambda)$ is also a weight and*

$$\dim V_\lambda = \dim V_{s_\alpha(\lambda)}.$$

In particular, if β is a root then also $s_\alpha(\beta)$ is a root.

Proof. The sum

$$U := \oplus_{i \in \mathbb{Z}} V_{\lambda + i\alpha}$$

is an $\mathfrak{sl}_2(\alpha)$-subrepresentation. Since $(\lambda + i\alpha)(H_\alpha) = \lambda(H_\alpha) + 2i$, the weight spaces $V_{\lambda+i\alpha}$ are also the H_α eigenspaces. Since in general the i-eigenspace and the $-i$ eigenspace for H_α have the same dimension, it follows from

$$s_\alpha(\lambda)(H_\alpha) = \lambda(H_\alpha) - \lambda(H_\alpha)\alpha(H_\alpha) = -\lambda(H_\alpha)$$

that

$$\dim V_\lambda = \dim V_{s_\alpha(\lambda)}.$$

□

Remark. Let β be a root not equal to α or $-\alpha$. Put $n, m \in \mathbb{Z}$ be maximal such that $\beta - m\alpha \in \Phi$ and $\beta + n\alpha \in \Phi$. Then the smallest $\mathfrak{sl}_2(\alpha)$-subrepresentation of $\mathfrak{g}$ containing $\mathfrak{g}_\beta$ is $\oplus_{i\in\mathbb{Z}, -m\leq i\leq n}\mathfrak{g}_{\beta+i\alpha}$ of dimension $n+m+1$. So H_α acts on $\mathfrak{g}_\beta$ by the scalar $m-n$, i.e. $\beta(H_\alpha) = m-n$.

The reflection s_α is also an isometry.

Lemma 4.4. *The involution s_α is an isometry with respect to the bilinear form $(\cdot,\cdot)$.*

Proof. For $\lambda, \mu \in \mathfrak{h}^*$ we get

$$\begin{aligned}(s_\alpha(\lambda), s_\alpha(\mu)) &= (\lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha, \mu - \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha) \\ &= (\lambda,\mu) - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}(\alpha,\mu) - \frac{2(\mu,\alpha)}{(\alpha,\alpha)}(\lambda,\alpha) + \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\frac{2(\mu,\alpha)}{(\alpha,\alpha)}(\alpha,\alpha) \\ &= (\lambda,\mu).\end{aligned}$$

□

Define the *Weyl group* as the group of isometries of $\mathfrak{h}^*$ generated by the reflections s_α, $\alpha \in \Phi$. In particular, it permutes the finite set of roots Φ. If V is any $\mathfrak{g}$-representation then all the weights are also permuted by the Weyl group.

Proposition 4.5. *The Weyl group of a semi-simple complex Lie algebra is finite.*

Proof. The roots form a generating set of the vector space $\mathfrak{h}^*$, by Proposition 4.1, so we know $w \in W$ when we know what w does with a root α. But we know that $w(\alpha)$ is also a root, so there are only finitely many possibilities and so W is finite. □

4.6. Integral weights. Define the *integral weights* to be the elements of

$$\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^*;\ \forall \alpha \in \Phi :\ \lambda(H_\alpha) \in \mathbb{Z}\}$$

It contains all the weights of all the $\mathfrak{g}$-representations. The Weyl group acts on it. It contains the *root lattice*, the integral span of all the roots.

Remark. It can be shown that $\mathfrak{g}$ is simple if and only if $\mathfrak{h}^*$ is a simple representation of the Weyl group.

Let V be a complex vector space (like our $\mathfrak{h}$) with a finite subset $R \subset V$ not containing 0 (the roots). It is called a (reduced) irreducible root system if the following conditions are satisfied:

(i) For every $\alpha \in R$ there exists a $\alpha^\vee \in V^*$ such that $\alpha^\vee(\alpha) = 2$ and the involution $s_{\alpha,\alpha^\vee}(v) := v - \alpha^\vee(v)\alpha$ stabilizes R.

(ii) For every $\alpha, \beta \in R$ we have $\alpha^\vee(\beta) \in \mathbb{Z}$.

(iii) If $\alpha \in R$ and n is not 1 or -1, then $n\alpha \notin R$.

(iv) The group W generated by $s_{\alpha,\alpha^\vee}$ is finite.

(v) The representation V for the group W is irreducible.

Such systems of roots can be classified. Any simple complex Lie algebra gives rise to such a root system, and vice versa *any such root system gives rise to a unique simple Lie complex Lie algebra.*

We will no have time to do this here.

4.7. Simple roots. There is a choice of *simple roots* Σ, having the following properties (without proof). They form a complex basis of $\mathfrak{h}^*$ and for any root $\alpha \in \Phi$ there exist (unique) integers n_β, $\beta \in \Sigma$ such that

$$\alpha = \sum_{\beta \in \Sigma} n_\beta \beta.$$

These integers all have the same sign. The root is called *positive* if and only if all n_β's are non-negative. Write Φ^+ for the collection of positive roots.

The co-roots associated to Σ form a basis of $\mathfrak{h}$.

The Lie subalgebras $\mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^\circ \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$ are nilpotent Lie algebras, whose elements all act nilpotently on any $\mathfrak{g}$-representation (but not on all $\mathfrak{n}$-representations). They are not contained in larger nilpotent Lie sub-algebras.

The Lie subalgebras $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}$ and $\mathfrak{b}^\circ := \mathfrak{h} + \mathfrak{n}^\circ$ are solvable Lie-subalgebras, not contained in any strictly larger solvable Lie sub-algebra of $\mathfrak{g}$. The nilpotent radical of $\mathfrak{b}$ is $\mathfrak{n}$ and the nilpotent radical of $\mathfrak{b}^\circ$ is $\mathfrak{n}^\circ$.

Since Σ forms a basis of $\mathfrak{h}^*$ any $\lambda \in \mathfrak{h}^*$ can be uniquely written as

$$\lambda = \sum_{\beta \in \Sigma} c_\beta \beta,$$

with $c_\beta \in \mathbb{C}$. We get a *partial ordering* on $\mathfrak{h}^*$ by defining $\lambda \leq \mu$ if and only if

$$\mu - \lambda = \sum_{\beta \in \Sigma} n_\beta \beta,$$

where n_β are all non-negative integers.

Make a choice of simple roots $\Sigma = \{\alpha_1, \ldots, \alpha_r\}$. Put $H_i = H_{\alpha_i}$. Then $H_1, \ldots, H_r$ is a basis for $\mathfrak{h}$. The integral $r \times r$ matrix A with coefficients

$$A_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \frac{2(\alpha_i^*, \alpha_j^*)}{(\alpha_i^*, \alpha_i^*)} = (H_i, \alpha_j^*) = \alpha_j(H_i) \in \mathbb{Z}$$

is called the *Cartan-matrix*.

There is another basis $\omega_1, \ldots, \omega_r$ of $\mathfrak{h}^*$, called *fundamental weights*, that is dual to $H_1, \ldots, H_r$ in the sense that

$$\omega_i(H_j) = \frac{2(\omega_i, \alpha_j)}{((\alpha_j, \alpha_j)} = \delta_{ij}.$$

Then for any $\lambda \in \mathfrak{h}^*$ we get the decomposition

$$\lambda = \sum_{i=1}^{r} \lambda(H_i)\omega_i.$$

In particular,

$$\alpha_j = \sum_{i=1}^{r} \alpha_j(H_i)\omega_i = \sum_{i=1}^{r} A_{ij}\omega_i.$$

So the Cartan matrix is the change-of-basis matrix.

The *integral weight lattice* $\mathfrak{h}^*_{\mathbb{Z}}$ can then be identified with the free abelian group

$$\Lambda := \oplus_{i=1}^{r} \mathbb{Z}\omega_i \subset \mathfrak{h}^*$$

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