LIE ALGEBRAS AND LIE GROUPS
MAT6633

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References

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1. Lie algebras

Let k be a field. A Lie algebra over k is a k vector-space $\mathfrak{g}$ together with a k-bilinear product,

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : x, y \mapsto [x, y],$$

called the (Lie-)bracket, such that $[x, x] = 0$ for all $x \in \mathfrak{g}$ and the the Jacobi-identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

holds for all $x, y, z \in \mathfrak{g}$.

We shall always assume that $\mathfrak{g}$ is finite dimensional as a vector space and that the characteristic of k is zero.

Let V be a (finite dimensional) vector-space over k. Then

$$\mathrm{End}_k(V) := \{\phi : V \to V; \quad \phi \text{ is } k\text{-linear}\}$$

is of course an associative unitary algebra over k, but it is also a Lie algebra if we define the bracket by

$$[X, Y] := XY - YX,$$

where $X, Y \in \mathrm{End}_k(V)$. The Jacobi-identity is checked by writing it out and using the associativity of the usual composition of endomorphisms.

A morphism of Lie algebras over k is a linear map between Lie algebras over k

$$\phi : \mathfrak{g} \to \mathfrak{h}$$

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such that $\phi([x,y]) = [\phi(x),\phi(y)]$. The collection of automorphisms is a group $\mathrm{Aut}(\mathfrak{g})$. In particular, any map $\sigma : \mathfrak{g} \to \mathfrak{g}$ is in $\mathrm{Aut}(\mathfrak{g})$ if and only if it is a k-linear isomorphism such that $\sigma([X,Y]) = [\sigma(X),\sigma(Y)]$, for all $X,Y \in \mathfrak{g}$.

A *Lie sub-algebra* (over k) is a sub vector-space $\mathfrak{h} \subset \mathfrak{g}$ such that if $x,y \in \mathfrak{h}$ then $[x,y] \in \mathfrak{h}$. A Lie sub-algebra $\mathfrak{a}$ is called an *ideal* if even $[x,y] \in \mathfrak{a}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$; we write $\mathfrak{a} \lhd \mathfrak{g}$.

The kernel of a morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras over k is an ideal in $\mathfrak{g}$, and the image is a Lie sub-algebra of $\mathfrak{h}$. If $\mathfrak{a} \lhd \mathfrak{g}$ then the quotient space $\mathfrak{g}/\mathfrak{a}$ is a Lie algebra with bracket $[\overline{x},\overline{y}] := \overline{[x,y]}$ and the natural map $\nu : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a} : X \to \overline{X}$ is a surjective Lie algebra morphism.

Like in group theory, ring theory or module theory we have the usual isomorphism theorems. For every morphism of Lie algebras $\mathfrak{g}/\,\mathrm{Ker} \simeq \mathrm{Im}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then $\mathfrak{a}+\mathfrak{b}$ and $\mathfrak{a}\cap\mathfrak{b}$ are also ideals, and $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$. There is a correspondence between the ideals (Lie sub-algebras) of $\mathfrak{g}/\mathfrak{a}$ and the ideals (Lie sub-algebras) of $\mathfrak{g}$ containing $\mathfrak{a}$.

If A and B are k-subspaces of $\mathfrak{g}$, define $[A,B]$ to be the k-subspace spanned by the set $\{[a,b]; a \in A, b \in B\}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then $[\mathfrak{a},\mathfrak{b}]$ is also an ideal.

A *derivation* D of a Lie algebra $\mathfrak{g}$ is a k-linear map $D : \mathfrak{g} \to \mathfrak{g}$ such that

$$D([x,y]) = [D(x),y] + [x,D(y)].$$

The collection $\mathrm{Der}(\mathfrak{g})$ of all derivations of a Lie algebra $\mathfrak{g}$ is a Lie sub-algebra of $\mathrm{End}_k(\mathfrak{g})$ is a Lie sub-algebra, since if D_1 and D_2 are derivations then also $[D_1,D_2] := D_1 \circ D_2 - D_2 \circ D_1$ is a derivation, as is checked easily.

Any $X \in \mathfrak{g}$ gives rise to the derivation $\mathrm{ad}(X) : Y \mapsto [X,Y]$ (called *inner derivation*). The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a morphism of Lie algebras over k. Its kernel is the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$:

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{g} : [X,Y] = 0\}.$$

The image of ad is even an ideal in $\mathrm{Der}(\mathfrak{g})$, since for any derivation D and $X \in \mathfrak{g}$ we have

$$[D, \mathrm{ad}(X)] = \mathrm{ad}(D(X)).$$

A derivation D is called *nilpotent* if there is an $n \geq 1$ such that $D^n = D \circ D \circ D \circ \ldots \circ D = 0$. Define the Lie algebra automorphism $\exp(D) \in \mathrm{Aut}(\mathfrak{g})$ by

$$\exp(D) = \sum_{i=0}^{\infty} \frac{1}{i!} D^i$$

which makes sense since the field is of characteristic zero and $D^n = 0$ if n is big enough. Then $\exp(tD) \circ \exp(sD) = \exp((s+t)D)$, for any $s,t \in k$, and in particular $\exp(D)^{-1} = \exp(-D)$. If $k = \mathbb{R}$ or $k = \mathbb{C}$ then the definition of $\exp(D)$ makes even sense for any endomorphism, since the limit exists, and $\exp(D) \in \mathrm{Aut}(\mathfrak{g})$ if $D \in \mathrm{Der}(\mathfrak{g})$.

Proof. Let $X,Y \in \mathfrak{g}$. By induction on r we prove that

$$D^r([X,Y]) = \sum_{i=0}^{r} \binom{r}{i} [D^i(X), D^{r-i}(Y)].$$

For $r = 1$ it is true by the definition of D being a derivation. Supposing true for r then

$$\begin{aligned} DD^r([X,Y]) &= \sum_{i=0}^{r} \binom{r}{i} D([D^i(X), D^{r-i}(Y)]) \\ &= \sum_{i=0}^{r} \binom{r}{i} ([D^{i+1}(X), D^{r-i}(Y)] + [D^i(X), D^{r+1-i}(Y)) \\ &= \sum_{i=0}^{r+1} \binom{r+1}{i} ([D^i(X), D^{r+1-i}(Y)]). \end{aligned}$$

And so

$$\begin{aligned} [\exp(D)(X), \exp(D)(Y) = &= [\sum_i \frac{1}{i!} D^i(X), \sum_j \frac{1}{j!} D^j(Y)] \\ &= \sum_r \sum_{i+j=r} \frac{1}{i!j!} [D^i(X), D^i(Y)] \\ &= \sum_r \frac{1}{r!} D^r([X,Y])] \\ &= \exp(D)([X,y]). \end{aligned}$$

□

A subalgebra is an ideal if and only if it is stable under all inner derivations. We shall say that $\mathfrak{a}$ is a *characteristic ideal* and write $\mathfrak{a} \blacktriangleleft \mathfrak{g}$ if it is even stable under all derivations. For example, the center of a Lie algebra is characteristic $\mathfrak{z}(\mathfrak{g}) \blacktriangleleft \mathfrak{z}$. The *derived subalgebra* is also $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] \blacktriangleleft \mathfrak{g}$.

That $\mathfrak{a} \lhd \mathfrak{b} \lhd \mathfrak{g}$ does not imply that $\mathfrak{a} \lhd \mathfrak{g}$. But $\mathfrak{a} \blacktriangleleft \mathfrak{b} \lhd \mathfrak{g}$ does imply that $\mathfrak{a} \lhd \mathfrak{g}$, and $\mathfrak{a} \blacktriangleleft \mathfrak{b} \blacktriangleleft \mathfrak{g}$ does imply that $\mathfrak{a} \blacktriangleleft \mathfrak{g}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are characteristic ideals, then $[\mathfrak{a}, \mathfrak{b}]$ is also a characteristic ideal.

So if we define $\mathfrak{g}^{(n)}$ by $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$, then $\mathfrak{g}^{(n)} \blacktriangleleft \mathfrak{g}$. So if we define $\mathfrak{g}_n$ by $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$, then $\mathfrak{g}_n \blacktriangleleft \mathfrak{g}$. We have $\mathfrak{g}_n \subseteq \mathfrak{g}^{(n)}$.

Let $\mathfrak{a}$ be any linear subspace such that $\mathfrak{g}' \subseteq \mathfrak{a} \subseteq \mathfrak{g}$. Then $\mathfrak{a} \lhd \mathfrak{g}$, since we even have $[X, Y] \in [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}$ for any X and Y.

A Lie algebra over k is called *solvable* if for some $n > 0$ $\mathfrak{g}^{(n)} = 0$. A Lie algebra over k is called *nilpotent* if for some $n > 0$ $\mathfrak{g}_n = 0$. So if $\mathfrak{g}$ is nilpotent it is also solvable. We have $\mathfrak{g}_n = 0$ if and only if for all $X_1, X_2, \ldots, X_n, Y \in \mathfrak{g}$ we have

$$\mathrm{ad}(X_1)\,\mathrm{ad}(X_2) \ldots \mathrm{ad}(X_n)(Y) = [X_1, [X_2, [X_3, \ldots [X_n, Y]]] \ldots] = 0.$$

Let $\mathfrak{a} \lhd \mathfrak{g}$. Then $\mathfrak{g}$ is solvable if and only if both $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are solvable. And if $\mathfrak{g}$ is nilpotent then $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ both are nilpotent.

If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals, then $\mathfrak{a} + \mathfrak{b}$ is also a solvable ideal.

Proof. Since $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{b}$ are solvable, also $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b}) \simeq (\mathfrak{a} + \mathfrak{b})/\mathfrak{a}$ is solvable. Since $\mathfrak{a}$ is also solvable, it follows that $\mathfrak{a} + \mathfrak{b}$ is solvable. □

If follows from the finite dimensionality of $\mathfrak{g}$ there is a biggest solvable ideal containing all other solvable ideals, called the *(solvable) radical* $\mathrm{rad}\mathfrak{g}$ of $\mathfrak{g}$. (It is in fact a characteristic ideal, but the proof is not direct.)

A Lie algebra is called *semi-simple* if its radical is 0. A Lie algebra is called *simple* if it is non-abelian and has no non-trivial ideals.

Example 1.1. Let V be an n-dimensional vector-space over k. A *(complete) flag* is a sequence of sub vector-spaces

$$0 = V_0 \subset V_1 \subset V_2, \ldots, V_{n-1} \subset V_n = V$$

with $\dim_K V_i = i$.

Then

$$\{X \in \mathrm{End}_k V; \forall 1 \leq i \leq n : XV_i \subseteq V_i\}$$

is a solvable Lie algebra over k and

$$\{X \in \mathrm{End}_k V; \forall 1 \leq i \leq n : XV_i \subseteq V_{i-1}\}$$

is a nilpotent Lie algebra.

Example 1.2. Let

$$\mathfrak{sl}_2(k) := \{ \begin{pmatrix} c & a \\ b & -a \end{pmatrix}; a, b, x \in k\}$$

be the Lie algebra of 2×2-matrices with coefficients in k and trace 0. It has basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F := \{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H := \{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with brackets $[E, F] = H$, $[H, E] = 2E$ and $[H, F] = -2F$. It is a simple Lie algebra.

Proof. Suppose $\mathfrak{a} \lhd \mathfrak{sl}_2(k)$ is a non-zero ideal. Suppose $A = aE + bF + cH \in \mathfrak{a}$ and $a \neq 0$. Then also $[E, aE+bF+cH] = bH - 2cE \in \mathfrak{a}$ and $[E, bH - 2cE] = -2bE \in \mathfrak{a}$ and so $E \in \mathfrak{a}$ and $[E, F] = H \in \mathfrak{a}$ and $[H, F] = 2F \in \mathfrak{a}$. So $\mathfrak{a} = \mathfrak{sl}_2(k)$. Similarly if $b \neq 0$ or if $A = cH$. □

1.1. **Representations.** The notion of representation of a Lie algebra plays the same role as an module over a ring. Let $k \subset K$ be a field extension and $\mathfrak{g}$ a Lie-algebra over k. A (finite dimensional) *K-representation* for $\mathfrak{g}$ on V is a morphism of Lie algebras over k:

$$F : \mathfrak{g} \to \mathrm{End}_K(V),$$

where V is (finite dimensional) vector space over K. We then use often the short-hand notation

$$X \cdot v = Xv := F(X)(v),$$

for $v \in V$ and $X \in \mathfrak{g}$. Then in particular

$$X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = [X, Y] \cdot v,$$

for $v \in V$ and $X, Y \in \mathfrak{g}$. We remark that although XY is not defined in the Lie algebra, the linear operator $F(X)F(Y$ is defined, and we define $XY \cdot v$ as $X \cdot (Y \cdot v)$.

On the other hand, suppose we have a product

$$\mathfrak{g} \times V \to V : (X, v) \mapsto X \cdot v$$

which is K-linear in v, and k-linear in X such that

$$X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = [X, Y] \cdot v,$$

then we get a representation $F : \mathfrak{g} \to \mathrm{End}_K(V)$ if we define $F(X)(v) := X \cdot v$. We shall then also say that V is a *K-representation* (where the action is defined).

If $k = K$ we just say representation, instead of k-representation.

A sub K-representation $U \subset V$ is a linear subspace such that for all $X \in \mathfrak{g}$ and $u \in U$ we have $X \cdot u \in U$. If U_1 and U_2 are sub K-representations, then also $U_1 + U_2$ and $U_1 \cap U_2$ are sub K-representations.

If $U \subset V$ is a sub K-representation then V/U becomes a K-representation when we define $X \cdot \overline{v} := \overline{X \cdot v}$, where $X \in \mathfrak{g}$ and $\overline{v} = v + U \in V/U$. Using the natural map $\nu : V \to V/U$ we get a correspondence between the K-sub-representations of V/U and the K-sub-representations of V containing U.

One special representation is the adjoint representation

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_k(\mathfrak{g}) : X \mapsto \mathrm{ad}(X).$$

A sub-representation of $\mathfrak{g}$ is then the same notion as an ideal. The Lie algebra $\mathrm{Der}_k(\mathfrak{g})$ also has a representation on $\mathfrak{g}$, by $D \cdot X := D(X)$. Then a sub-representation of $\mathfrak{g}$ for the Lie algebra $\mathrm{Der}_k(\mathfrak{g})$ is the same as a characteristic ideal.

Suppose $V = Kv$ is a one-dimensional K-representation, then there is a function $\lambda : \mathfrak{g} \to K$ defined by

$$X \cdot v = \lambda(X)v$$

such that λ is K-linear and $\lambda([X, Y]) = 0$ since

$$[X, Y] \cdot v = X \cdot Y \cdot v - Y \cdot X \cdot v = \lambda(X)\lambda(Y)v - \lambda(Y)\lambda(X)v = 0.$$

So every bracket acts trivially on any K-representation. In fact, λ does not depend on the choice of basis element of V. Inversely, any such function λ gives rise to one-dimensional K-representation. So one-dimensional K-representations are in bijection with the k-linear maps $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to K$.

1.2. **Lie's theorem.** One of the important classical theorems of Sophus Lie concerns the representation theory of solvable Lie algebras.

Theorem 1.1 (Lie)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra over k. Let $k \subset K$ a field extension and V be a (finite dimensional) K-representation on V. Suppose that for every $X \in \mathfrak{g}$ all the eigenvalues of the linear maps $v \mapsto X \cdot v$ are in K (this is of course the case if K is algebraically closed).*

Then there is a complete flag

$$0 = V_0 \subset V_1 \subset V_2, \ldots, V_{n-1} \subset V_n = V$$

of sub K-representations.

Hence on an appropriate basis all matrices associated to the elements of $\mathfrak{g}$ are simultaneously represented by triangular matrices.

Proof. We start by proving that there is a one-dimensional sub K-representation V_1. Or in other words, there is a simultaneous common eigenvector $v_1 \in V$, i.e., $v_1 \neq 0$ and for all $Z \in \mathfrak{g}$ there exists a $\lambda(Z) \in K$ such that $Z \cdot v_1 = \lambda(Z)v_1$. Then $V_1 = Kv_1$. We shall use induction on $\dim_k \mathfrak{g}$.

If $\mathfrak{g} = kX$ is one-dimensional then the eigenvalues of X acting K-linearly on V are all contained in K, so by linear algebra there is indeed an eigenvector v for X with some eigenvalue λ. So for all $cX \in \mathfrak{g}$ $(c \in k)$ we have $cX \cdot v = c\lambda v$.

Assume that the dimension of $\mathfrak{g}$ is ≥ 2 and that the result is true for solvable Lie algebras of lower dimension. Since $\mathfrak{g}$ is solvable, the derived Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is unequal to $\mathfrak{g}$. Take any k-linear subspace $\mathfrak{a}$ of codimension one in $\mathfrak{g}$ such that $\mathfrak{a} \supseteq \mathfrak{g}'$. Then $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, and therefore is also a solvable Lie algebra. Take any $X \notin \mathfrak{a}$, then $kX \oplus \mathfrak{a} = \mathfrak{g}$ as vector spaces. By induction there is a $w \in V$ $(w \neq 0)$ and a function $\lambda : \mathfrak{a} \to K$ such that for all $Y \in \mathfrak{a}$

$$Yw = \lambda(Y)w.$$

Write now $V_\lambda = \{u \in V; \forall Y \in \mathfrak{a} : Yu = \lambda(Z)u\}$. It is a non-zero linear subspace, since it contains w, and is $\mathfrak{a}$-stable. We want to show that V_λ is even a K-subrepresentation of V for the Lie algebra $\mathfrak{g}$.

If $Z = cX + Y$, $Y \in \mathfrak{a}$ and $u \in V_\lambda$ we have

$$Z \cdot u = cX \cdot u + Y \cdot u = cX \cdot u + \lambda(Y)u.$$

So it suffices to prove that $X \cdot u \in V_\lambda$. Now

$$Y \cdot (X \cdot u) = [Y, X] \cdot u + X \cdot Y \cdot u = \lambda([Y, X])u + X \cdot (\lambda(Y)u) = \lambda([Y, X])u + \lambda(Y)(X \cdot u),$$

hence it suffices to prove that $\lambda([Y, X]) = 0$ for all $Y \in \mathfrak{a}$.

For this we use the following argument. Define inductively $e_0 = w$ and $e_{i+1} = X \cdot e_i$ for $i \geq 0$. Put U_i $(i \geq 0)$ for the K-linear subspace of V spanned by $e_0, \ldots, e_i$, and put $U_{-1} = 0$. Let m be minimal such that $X \cdot e_m = e_{m+1} \in U_m$, then necessarily $U_r = U_m$ for all $r \geq m$. Put $U = U_m$, then $\dim_K U = m + 1$ and U is X-stable.

We claim that $Y \cdot e_i - \lambda(Y)e_i \in U_{i-1}$ for any $Y \in \mathfrak{a}$. This is certainly true for $i = 0$ by the choice of $e_0 = w$. Suppose it is true for $i \geq 0$, then

$$\begin{aligned} Y \cdot e_{i+1} - \lambda(Y)e_{i+1} &= Y \cdot (X \cdot e_i) - \lambda(Y)X \cdot e_i \\ &= [Y, X] \cdot e_i + X \cdot (Y \cdot e_i) - X \cdot (\lambda(Y)e_i) \in U_i \end{aligned}$$

since $[Y, X]e_i \in U_i$ by induction and also

$$X \cdot (Y \cdot e_i) - X \cdot (\lambda(Y)e_i) = X \cdot (Y \cdot e_i - \lambda(Y)e_i) \in X \cdot U_{i-1} \subseteq U_i.$$

So the claim follows and U is K-sub-representation for $\mathfrak{g}$ and we get a Lie algebra homomorphism $\tilde{F} : \mathfrak{g} \to \mathrm{End}_K(U)$ such that $\tilde{F}(X)(u) = X \cdot u$ (for $X \in \mathfrak{g}$ and $u \in U$).

We calculate the trace of $\tilde{F}([X, Y])$ acting on U in two ways. Since $[X, Y] \in \mathfrak{a}$, we can calculate this trace using the basis $e_0, \ldots e_m$ and the claim, and we get the value $(m + 1)\lambda([X, Y])$. On the other hand the trace of a commutator is always 0 and $\mathrm{tr}(\tilde{F}([X, Y]) = \mathrm{tr}(\tilde{F}(X)\tilde{F}(Y)) - \mathrm{tr}(\tilde{F}(Y)\tilde{F}(X)) = 0$. Hence the trace of $\tilde{F}([X, Y])$ acting on U is $0 = (m + 1)\lambda([X, Y])$. We conclude that $\lambda([X, Y]) = 0$.

So indeed V_λ is $\mathfrak{g}$-stable. Now let $v_1 \in V_\lambda$ be an eigenvector for X, then it is a simultaneous eigenvector for $\mathfrak{g}$ acting on V. So there is a function $\lambda_1 : \mathfrak{g} \to K$ such that $X \cdot v_1 = \lambda(X)v_1$.

Now we shall prove the full version of Lie's theorem by induction on the dimension of V. If $\dim_K V = 1$ we have nothing left to do. Suppose $\dim_K V > 1$. We just showed there is a one-dimensional K-sub-representation V_1 for $\mathfrak{g}$.

Now we consider the quotient K-representation V/V_1 having lower dimension. So we can use induction and the correspondence at the end of last sub-section to get we get a flag $0 = V_0 \subset V_1 \subset V_2 \subset \ldots V_n = V$ of K-sub-representations of V such that $0 = V_1/V_1 \subset V_2/V_1 \subset \ldots V_n/V_1$ is a flag of K-sub-representations of V/V_1. We finished the proof of Lie's theorem. ☐

1.3. **Engel's theorem.** The following theorem due to Friedrich Engel deals with the situation when all the elements of $\mathfrak{g}$ act nilpotently in some representation V.

Theorem 1.2 (Engel)**.** *Let $\mathfrak{g}$ be a Lie algebra over k. Let $k \subset K$ a field extension and V be a (finite dimensional) K-representation. Suppose that every $X \in \mathfrak{g}$ acts nilpotently on V.*

Then there is a complete flag

$$0 = V_0 \subset V_1 \subset V_2, \ldots, V_{n-1} \subset V_n = V$$

of sub K-representations such that even for all $X \in \mathfrak{g}$ and $1 \leq i \leq n$ we have

$$X \cdot V_i \subseteq V_{i-1}.$$

So there is a K-basis $v_1, \ldots, v_n$ of V such that for all $X \in \mathfrak{g}$: $X \cdot v_1 = 0$ and for all $i \geq 2$: $X \cdot v_i \in Kv_1 \oplus \ldots \oplus Kv_{i-1}$. Or on this basis all $X \in \mathfrak{g}$ are simultaneously represented by strictly upper-triangular matrices.

In particular, the image of $\mathfrak{g}$ inside $\mathrm{End}_K(V)$ is a nilpotent Lie algebra.

Proof. First a preparation. Replacing $\mathfrak{g}$ by its image in $\mathrm{End}_K(V)$ we can assume that $\mathfrak{g} \subset \mathrm{End}_K(V)$ is a Lie sub algebra over k. Also $E = \mathrm{End}_K(V)$ is a K-vector space. For any $M \in \mathrm{End}_K(V)$ let $L(M) \in \mathrm{End}_K(E)$ be left multiplication on E and let $R(M) \in \mathrm{End}_K(E)$ be right multiplication, hence for any $N \in \mathrm{End}_K(V)$ we have $L(M)(N) = MN$ and $R(M)(N) = NM$ and for any M_1, M_2 we have that $L(M_1)$ and $R(M_2)$ commute. So

$$(L(M) - R(M))^r = \sum_{i=0}^{r} \binom{r}{i} L(M)^i (-R(M))^{r-i} = \sum_{i=0}^{r} \binom{r}{i} L(M^i) R((-M)^{r-i}).$$

If $M \in \mathfrak{g}$ then $M \in \mathrm{End}_K(V)$ is nilpotent and hence also $L(M)$ and $R(M)$ are nilpotent and therefore $L(M) - R(M))$ is nilpotent, by the equation above. If $X, Y \in \mathfrak{g}$ then $\mathrm{ad}(X)(Y) = [X, Y] = L(X)(Y) - R(X)(Y) = \{L(X) - R(X)\}(Y)$. Hence $\mathrm{ad}(X) \in \mathrm{End}_k \mathfrak{g}$ is nilpotent for every $X \in \mathfrak{g}$.

By induction on $\dim_k \mathfrak{g}$ we shall show that that for any non-zero K-representation V for $\mathfrak{g}$, where all $X \in \mathfrak{g}$ act nilpotently on V, there always exist a non-zero vector $v \in V$ such that $X \cdot v = 0$ for all $X \in v$.

If $\mathfrak{g} = kX$ is one-dimensional and X acts nilpotently on $V \neq 0$, then 0 is the only eigenvalue for X acting K-linearly on V, and therefore by linear algebra there is an eigenvector v with zero eigenvalue 0, i.e., $tXv = 0$ for all $t \in k$ and hence for all $tX \in \mathfrak{g}$. We assume that the result is true for Lie algebras of smaller dimension than $\mathfrak{g}$.

Let now $\mathfrak{a} \subset \mathfrak{g}$ be a maximal proper Lie sub-algebra of $\mathfrak{g}$. Then for any $Y \in \mathfrak{a}$ and $Z \in \mathfrak{a}$ we have $\mathrm{ad}(Y)(Z) \in \mathfrak{a}$, so we get a representation of $\mathfrak{a}$ on the k-vector space $\mathfrak{g}/\mathfrak{a}$, and any element $Y \in \mathfrak{a}$ acts nilpotently on $\mathfrak{g}/\mathfrak{a}$. By induction there is an $X \in \mathfrak{g}$, $X \notin \mathfrak{a}$ such that $\mathrm{ad}(Y)(\overline{X}) = 0$ for all $Y \in \mathfrak{a}$, or in other words $[Y, X] \in \mathfrak{a}$ for all $Y \in \mathfrak{a}$. It follows that $kX \oplus \mathfrak{a}$ is a strictly larger Lie sub-algebra than $\mathfrak{a}$. By the maximality hypothesis, it follows that $\mathfrak{g} = kX \oplus \mathfrak{a}$ and $\mathfrak{a} \lhd \mathfrak{g}$.

Returning to V, let $V_0 := \{v \in V; \forall Y \in \mathfrak{a} : Y \cdot v = 0\}$. By the induction hypothesis it is non-zero. Let $v \in V_0$ and $Y \in \mathfrak{a}$ then

$$YXv = [Y, X]v + XYv = 0 + 0 = 0.$$

Hence V_0 is also stable by X, hence stable by $\mathfrak{g}$. Now take any non-zero $v \in V_0$ that is an eigenvector of X with eigenvalue 0, then $Zv = 0$ for all $Z \in \mathfrak{g}$. This ends the induction proof.

We finish the proof of Engel's theorem by induction on the dimension of V. If $\dim_K V = 1$, then there exists a non-zero v such that $Zv = 0$ for all $Z \in \mathfrak{g}$. Then $\{v\}$ is the K-basis of Engel's theorem. We assume Engel's theorem to be true for any representation that has smaller dimension than V. Let $v_1 \neq 0$ be as above, such that $Zv_1 = 0$ for all $Z \in \mathfrak{g}$. Then we get a $\mathfrak{g}$ representation on the quotient space V/Kv_1, such that any $Z \in \mathfrak{g}$ acts nilpotently. So we can use the induction hypothesis and conclude there are $v_2, \ldots, v_n$ such that for all $Z \in \mathfrak{g}$ we have that $Z\overline{v_2} = 0$ and for $i \geq 3$ that $Z\overline{v_i} \in K\overline{x_2} \oplus \ldots \oplus \overline{v_{i-1}}$. Or for all $i \geq 2$ that $Zv_i \in Kv_1 \oplus \ldots \oplus Kv_{i-1}$. This proves Engel's theorem. □

We apply Engel's theorem using the adjoint representation.

Corollary 1.1. *Let* $\mathfrak{g}$ *be a Lie algebra. Then* $\mathfrak{g}$ *is a nilpotent Lie algebra if and only if* $\mathrm{ad}(X)$ *is nilpotent for all* $X \in \mathfrak{g}$.

Proof. (i) Suppose that $\mathfrak{g}$ is nilpotent and that $\mathfrak{g}_m = 0$. Then in particular for all $Y \in \mathfrak{g}$ we have

$$ad(X)^m(Y) = [X, [X, [X, \ldots [X, [X, Y]] \ldots] = 0,$$

with m terms X. Hence $\mathrm{ad}(X)$ acts nilpotently on $\mathfrak{g}$.

(ii) Suppose $\mathrm{ad}(X)$ is nilpotent for all $X \in \mathfrak{g}$. Then Engel's theorem implies that $\mathrm{ad}(\mathfrak{g})$ is contained in a nilpotent Lie algebra, hence is a nilpotent Lie algebra itself. The kernel of ad is the center $\mathfrak{z}$ of $\mathfrak{g}$, so $\mathfrak{g}/\mathfrak{z} \simeq \mathrm{ad}(\mathfrak{g})$. There is an m such that $[\mathrm{ad}(\mathfrak{g}), [\mathrm{ad}(\mathfrak{g}), [\ldots, [\mathrm{ad}(\mathfrak{g}), \mathrm{ad}(\mathfrak{g})]] \ldots] = 0$ (with m terms $\mathrm{ad}(\mathfrak{g})$). So $[\mathfrak{g}, [\mathfrak{g}, [\ldots, [\mathfrak{g}, \mathfrak{g}]] \ldots] \subset \mathfrak{z}$ and bracketing with one more copy of $\mathfrak{g}$ gives

$$[\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, [\ldots, [\mathfrak{g}, \mathfrak{g}]] \ldots] \subset [\mathfrak{g}, \mathfrak{z}] = 0.$$

We conclude that $\mathfrak{g}$ is nilpotent as well. □

Now we apply Engel's theorem and its corollary in the situation of a solvable Lie algebra.

Corollary 1.2. *Let* $\mathfrak{g}$ *be a solvable Lie algebra. The collection*

$$\mathfrak{n} := \{X \in \mathfrak{g};\ \mathrm{ad}(X) \textit{ is nilpotent}\}$$

is the unique largest nilpotent ideal, called its nilpotent radical, i.e., all other nilpotent ideals are contained in it.

For any derivation D *of* $\mathfrak{g}$ *we get* $D(\mathfrak{g})$ *and* $[\mathfrak{g}, \mathfrak{g}]$ *are nilpotent and hence contained in* $\mathfrak{n}$.

Proof. Since $\mathfrak{g}$ is solvable, its image $\mathrm{ad}(\mathfrak{g})$ is a solvable Lie sub-algebra of $\mathrm{End}_k\,\mathfrak{g}$. Let $D \in \mathrm{End}_k(\mathfrak{g})$ be a derivation of $\mathfrak{g}$. Consider the k-linear subspace $\tilde{\mathfrak{g}} := kD + \mathrm{ad}(\mathfrak{g}) \subset \mathrm{End}_k(\mathfrak{g})$. Since $[D, \mathrm{ad}(X)] = \mathrm{ad}(D(X))$, it follows that $\tilde{\mathfrak{g}}$ is a Lie subalgebra of $\mathrm{End}_k(\mathfrak{g})$ with $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subset \mathrm{ad}(\mathfrak{g})$ and so it is solvable itself.

Let $k \subset K$, with K algebraically closed. Put $V = \mathfrak{g} \otimes_k K$, then

$$\tilde{\mathfrak{g}} \subseteq \mathrm{End}_k(\mathfrak{g}) \subseteq \mathrm{End}_K V,$$

and V is a K-representation of $\tilde{\mathfrak{g}}$.

We can apply Lie's theorem and get a complete flag

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$$

of K-representations of $\tilde{\mathfrak{g}}$.

Define functions $\lambda_i : \tilde{\mathfrak{g}} \to K$ as the common eigenvalue function on V_i/V_{i-1}: if $v \in V_i$ then for all $Z \in \tilde{\mathfrak{g}}$ we get $Z \cdot v - \lambda(Z)v \in V_{i-1}$. We know that any bracket acts trivially on any one-dimensional K-representation. So in particular, for any $X, Y \in \mathfrak{g}$ and i it holds that $\lambda_i(\mathrm{ad}([X,Y])) = 0 = \mathrm{ad}_i(\mathrm{ad}(D(X)))$.

Suppose $Z \in \tilde{\mathfrak{g}}$ is nilpotent and $Z^m = 0$ then necessarily for all i we have $\lambda_i(Z)^m = 0$, hence $\lambda_i(Z) = 0$. And conversely.

So in particular, for every $X, Y \in \mathfrak{g}$ and any derivation D we have $\mathrm{ad}([X,Y])$ and $\mathrm{ad}(D(X))$ are nilpotent. We define $\mu_i : \mathfrak{g} \to K$ by $\mu_i(X) := \lambda_i(\mathrm{ad}(X))$ which is a Lie algebra homomorphism and whose kernel is hence an ideal. It follows that $\mathfrak{n}$ as defined in the corollary is in fact an ideal of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$ and $D(\mathfrak{g})$ for any derivation. □

This last corollary allows us to define the *nilpotent radical* of any Lie algebra.

Corollary 1.3. *Let $\mathfrak{g}$ be a Lie algebra. Any nilpotent ideal is contained in a unique largest nilpotent ideal $\mathfrak{n} \lhd \mathfrak{g}$, called the nilpotent radical of $\mathfrak{g}$. It equals the nilpotent radical of* $\mathrm{rad}\mathfrak{g}$.

Proof. Let $\mathfrak{a}$ be any nilpotent ideal. Since it is solvable too, it is contained in the solvable radical $\mathrm{rad}(\mathfrak{g})$ and so in the largest nilpotent ideal $\mathfrak{n}$ of $\mathrm{rad}(\mathfrak{g})$. On the other hand if $X \in \mathfrak{g}$, then the restriction of $\mathrm{ad}(X)$ to $\mathrm{rad}(\mathfrak{g})$ is a derivation, and so by the previous corollary $[X, \mathfrak{n}] \subseteq [X, \mathrm{rad}\mathfrak{g}] \subseteq \mathfrak{n}$ and so $\mathfrak{n} \lhd \mathfrak{g}$. □

A Lie algebra whose nilpotent radical is zero is called *reductive.*

1.4. **Bilinear forms.** To any representation V of a Lie algebra we can associate a bilinear form on $\mathfrak{g}$. So let us discuss bilinear forms on a vector space first.

Let V be a finite-dimensional vector space over a field k and

$$C(\cdot,\cdot) : V \times V \to k$$

a bilinear form over k on $V \times V$. If $C(u,v) = C(v,u)$ for all $u, v \in V$, the bilinear form is called symmetric. If $C(u,v) = -C(v,u)$ it is called anti-symmetric. *We shall suppose that the form is either symmetric or anti-symmetric.*

Then at least

$$\mathfrak{a} = \{X \in \mathrm{End}_k(V); \forall u \in V, v \in V : C(X(u), v) + C(u, X(v)) = 0\}$$

is a Lie sub-algebra of $\mathrm{End}_k(V)$.

Proof. The bilinearity of $C(\cdot,\cdot)$ gives that $\mathfrak{a}$ is a linear subspace. Let $X, Y \in \mathfrak{a}$ and $u, v \in V$. Then

$$C((XY - YX)(u), v) = C(XY(u), v) - C(u, YX(v)) = -C(Y(u), X(v)) + C(Y(u), X(v)) = 0$$

so $[X, Y] \in \mathfrak{a}$. □

If $U \subset V$ is a linear subspace then we define $U^\perp$ by

$$U^\perp := \{v \in V; \forall u \in U : C(v, u) = 0\}.$$

The *radical* $\mathrm{rad}C$ of C is defined by

$$\mathrm{rad}C := \{v \in V; \forall u \in V : C(u, v) = 0\} = V^\perp.$$

We can restrict the bilinear form C to $U \times U$, say $C|_{U\times U}$. Then the radical of $C|_{U\times U}$ is just $U \cap U^\perp$.

The bilinear form C is called *non-degenerate* if its radical is 0. Let $v_1, \ldots, v_n$ be a basis of V. Put $C_{ij} := C(v_i, v_j)$ for $1 \le i \le n$ and $1 \le i \le n$. Then C is non-degenerate if and only if the matrix (C_{ij}) (called Gramm-matrix) is invertible.

Let $C(v, \cdot) \in V^* = \mathrm{Hom}_k(V, k)$ be defined by $C(v, \cdot)(u) := C(v, u)$, with $u \in V$. Then we get a k-linear map $V \to V^* : v \mapsto C(v, \cdot)$ with kernel $V^\perp = \mathrm{rad}C$; so this map is an isomorphism if and only if the radical of C is zero.

Lemma 1.1. *Suppose $C(\cdot,\cdot)$ is non-degenerate and $U \subset V$ is a linear subspace.*

Then $\dim V = \dim U + \dim U^\perp$. And $V = U + U^\perp$ if and only if $V = U \oplus U^\perp$ if and only if $U \cap U^\perp = 0$ if and only if the radical of $C|_{U\times U}$ is 0 if and only if the restriction of C to $U \times U$ is a non-degenerate bilinear form.

Proof. We define the linear map $\psi : V \to U^*$ by $[\psi(v)](u) := C(v, u)$ where $v \in V$ and $u \in U$. Its kernel is just $U^\perp$. We want to show that ψ is surjective.

Choose a vector space complement U' of U, i.e., $V = U \oplus U'$. For any linear form $\mu : U \to K$ define the linear form $\tilde{\mu} : V \to K$ by $\tilde{\mu}(u + u') := \mu(u)$, where $u \in U$ and $u' \in U'$.

Since $C(\cdot,\cdot)$ is non-degenerate, there is a unique $v \in V$ such that $\tilde{\mu} = C(v, \cdot)$. Then for this v:

$$[\psi(v)](u) = C(v, u) = \tilde{\mu}(u) = \mu(u),$$

for all $u \in U$. So ψ is surjective. So

$$\dim V - \dim U^\perp = \dim(V/U^\perp) = \dim U^* = \dim U$$

so

$$\dim V = \dim U + \dim U^\perp.$$

Now $U + U^\perp = V$ if and only if $\dim(U + U^\perp) = \dim V = \dim U + \dim U^\perp$ if and only if $U + U^\perp = U \oplus U^\perp$ if and only if $U \cap U^\perp = \mathrm{rad}C|_{U\times U} = 0$. □

Lemma 1.2. *Suppose $C(\cdot,\cdot)$ is non-degenerate and $v_1, \ldots, v_n$ a basis for V. Then there is a unique basis (called dual basis) $v'_1, \ldots, v'_n$ with the property that $C(v_i, v'_j) = \delta_{ij}$ (=Kronecker δ).*

If $v = \sum_i a_i v_i$ and $w = \sum_i b_i v'_i$ then $a_i = C(v, v'_i)$, $b_i = C(v_i, v)$ and $C(v, w) = \sum_i a_i b_i$.

Proof. Take $1 \leq i \leq n$ and put V_i be the linear subspace of V of co-dimension one spanned by all the basis vectors v_j where $j \neq i$. By the previous lemma $\dim V_i^\perp = 1$. On the other hand $\dim(V_i^\perp)^\perp = \dim V - 1$ and $V_i \subseteq (V_i^\perp)^\perp$, hence $V_i = (V_i^\perp)^\perp$. So there exists a $v_i' \in V_i^\perp$ such that $C(v_i, v_i') = 1$. So we found elements $v_1', \dots, v_n'$ such that $C(v_i, v_j') = \delta_{ij}$ (Kronecker δ). Suppose there is a linear relation $\sum_j c_j v_j' = 0$, then for all i we have

$$0 = C(v_i, 0) = C(v_i, \sum_j c_j v_j') = c_i,$$

so the relation is trivial and $v_1', \dots, v_n'$ is also a basis. □

1.5. **Bilinear forms on $\mathfrak{g}$.** Let $\mathfrak{g}$ be a Lie algebra over k and $F : \mathfrak{g} \to \mathrm{End}_k(V)$ any representation on the finite dimensional k-vector space V. Then it induces a symmetric bilinear form on $\mathfrak{g}$:

$$B_V(X, Y) := \mathrm{tr}(F(X)F(Y)).$$

It has the additional property that for all $X, Y, Z \in \mathfrak{g}$:

$$B_V([X, Z], Y) = B_V(X, [Z, Y]),$$

or

$$B_V([X, Z], Y) + B_V(Z, [X, Y]) = 0.$$

Since

$$\begin{aligned} \mathrm{tr}(F([X, Z])F(Y)) &= \mathrm{tr}(F(X)F(Z)F(Y)) - \mathrm{tr}(F(Z)F(X)F(Y)) \\ &= \mathrm{tr}(F(X)F(Z)F(Y)) - \mathrm{tr}(F(X)F(Y)F(Z)) \\ &= \mathrm{tr}(F(X)F([Z, Y])). \end{aligned}$$

Lemma 1.3. *If $U \subset V$ is a subrepresention for $\mathfrak{g}$ with quotient representation on V/U, then*

$$B_V(X, Y) = B_U(X, Y) + B_{V/U}(X, Y).$$

Proof. Let $v_1, \dots, v_n$ be a basis for V such that $v_1, \dots, v_m$ is a basis for U and $\overline{v_{m+1}}, \cdots, \overline{v_n}$ a basis for V/U. If $X \cdot Y \cdot v_j = c_{ij} v_i$ then the $n \times n$ matrix with coefficients c_{ij} $(1 \leq i \leq n, 1 \leq j \leq n)$ is the matrix associated to $v \to X \cdot Y \cdot v$ acting on V; the $m \times m$-matrix with coefficients c_{ij} $(1 \leq i \leq m, 1 \leq j \leq m$ is the matrix acting on U, and the $(n - m) \times (n - m)$-matrix with coefficients c_{ij} $(m + 1 \leq i \leq n, m + 1 \leq j \leq n)$ is the matrix on V/U. Now taking traces gives the lemma. □

Lemma 1.4. *Let $\mathfrak{g}$ be a Lie algebra with representation V.*

(i) Let $\mathfrak{a} \lhd \mathfrak{g}$ then $\mathfrak{a}^\perp \lhd \mathfrak{g}$ (the orthogonal space with respect to B_V). In particular, $\mathrm{rad} B_V = \mathfrak{g}^\perp \lhd \mathfrak{g}$.

(ii) Suppose $\mathfrak{a}$ is an ideal such that all $A \in \mathfrak{a}$ act nilpotently on V. Then $\mathfrak{a} \subseteq \mathrm{rad} B_V$.

Proof. (i) At least $\mathfrak{a}^\perp$ is a linear subspace of $\mathfrak{g}$. Let $H \in \mathfrak{a}^\perp$. For all $X \in \mathfrak{g}$ and $A \in \mathfrak{a}$ we have $[X, A] \in \mathfrak{a}$ and so

$$B_V([H, X], A) = B_V(H, [X, A]) = 0$$

so $[H, X] \in \mathfrak{a}^\perp$ and therefore $\mathfrak{a}^\perp \lhd \mathfrak{g}$.

(ii) We shall use induction on V. There is nothing to do if $\dim V = 0$. Let $\dim V > 0$. Let $V_0 = \{v \in V; \forall A \in \mathfrak{a} : A \cdot v = 0\}$. By Engel's theorem $V_0 \neq 0$. It is even a sub-representation for $\mathfrak{g}$, since if $X \in \mathfrak{g}$ and $v \in V_0$, then for all $A \in \mathfrak{a}$ we have $AXv = [A,X]v + XAv = 0$, since $[A,X] \in \mathfrak{a}$. So $AXv = 0$ for all $v \in V_0$ and so V_0 is a subrepresentation for $\mathfrak{g}$ and also $B_{V_0}(A,X) = 0$.

We get a quotient representation of $\mathfrak{g}$ on V/V_0 and any $A \in \mathfrak{a}$ still acts nilpotently on V/V_0 so by induction we can assume that $B_{V/V_0}(A,X) = 0$.

Since $B_V(A,X) = B_{V_0}(A,X) + B_{V/V_0}(A,X)$ we can conclude. □

1.6. **Killing's symmetric bilinear form on $\mathfrak{g}$.** Any finite dimensional Lie algebra $\mathfrak{g}$ over k has the adjoint representation on $\mathfrak{g}$ itself. Its associated k-bilinear form $B(\cdot,\cdot) = B_{\mathfrak{g}}(\cdot,\cdot)$ is called Killing's bilinear form (or the Killing form):

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)).$$

Lemma 1.5. *If $\mathfrak{a} \lhd \mathfrak{g}$ then the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a} \times \mathfrak{a}$ is the Killing form of $\mathfrak{a}$.*

Proof. We have $\mathfrak{a}$ is a subrepresentation of the (adjoint) representation $\mathfrak{g}$ and so $B(X,Y) = B_{\mathfrak{a}}(X,Y) + B_{\mathfrak{g}/\mathfrak{a}}(X,Y)$ and $B_{\mathfrak{g}/\mathfrak{a}}(X,Y)$ is 0 if $X, Y \in \mathfrak{a}$, since $\mathfrak{a}$ is an ideal. □

Proposition 1.1. *The nilpotent radical is contained in the radical of the Killing form.*

In particular, if $\mathfrak{g}$ is solvable then $[\mathfrak{g},\mathfrak{g}]$ is nilpotent and therefore contained in the radical of the Killing form.

Proof. Since an ideal $\mathfrak{a} \lhd \mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(A)$ is nilpotent for all $A \in \mathfrak{a}$, we get that every nilpotent ideal is contained in the radical of the Killing form, by Lemma 1.4, in particular the nilpotent radical is contained in the radical of the Killing form. □

The converse is also true.

Theorem 1.3 (Cartan's solvability criterion)**.** *Let $\mathfrak{g}$ be a finite dimensional k-algebra. Then $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{rad}B$.*

We shall give a proof in the next subsection.

Corollary 1.4. *Let $\mathfrak{g}$ be a finite dimensional k-algebra. Then $\mathrm{rad}B \subseteq \mathrm{rad}\mathfrak{g}$ (and $\mathrm{rad}B$ contains the nilpotent radical).*

Proof. It suffices to prove that the ideal $\mathfrak{a} := \mathrm{rad}B$ is solvable using Cartan's criterion. The restriction of the Killing form to $\mathfrak{a} \times \mathfrak{a}$ is the Killing form C on $\mathfrak{a}$. So for any $X, Y \in \mathfrak{a}$ we have $C(X,Y) = B(X,Y) = 0$ since X is in the radical of B. So we conclude using Cartan's criterion. □

Theorem 1.4 (Cartan)**.** *Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent.*

(i) The Killing form on $\mathfrak{g}$ is non-degenerate;

(ii) $\mathfrak{g}$ is semi-simple;

(iii) $\mathfrak{g}$ is the direct sum of simple ideals.

Proof. Suppose (i). The Killing form is non-degenerate hence $\mathfrak{g}$ does not have any non-zero nilpotent ideals and in particular the nilpotent radical of $\mathfrak{g}$ is zero. Since $[\mathrm{rad}(\mathfrak{g}), \mathrm{rad}(\mathfrak{g})]$ is nilpotent it follows that the radical of $\mathfrak{g}$ is abelian, say $\mathfrak{a}$. Let $X \in \mathfrak{g}$ and $A \in \mathfrak{a}$. For any $B \in \mathfrak{a}$ we have

$\operatorname{ad}(X)\operatorname{ad}(A)(B) = [X,[A,B]] = 0$, since $\mathfrak{a}$ is abelian. So $B_{\mathfrak{a}}(X,A) = 0$. On the other hand $B_{\mathfrak{g}/\mathfrak{a}}(X,A) = 0$ since $\mathfrak{a}$ is an ideal and $A \in \mathfrak{a}$. So

$$B(X,A) = B_{\mathfrak{a}}(X,A) + B_{\mathfrak{g}/\mathfrak{a}}(X,A) = 0.$$

So $A \in \operatorname{rad}(B) = 0$. We conclude that $\operatorname{rad}(\mathfrak{g}) = 0$ and $\mathfrak{g}$ is semi-simple. Hence (i) implies (ii).

Suppose still that the Killing form is non-degenerate. We will show (iii) holds. Let $\mathfrak{a} \lhd \mathfrak{g}$ be a minimal non-zero ideal. If $\mathfrak{a} = \mathfrak{g}$ then $\mathfrak{g}$ is simple and we are done. Then $[\mathfrak{a},\mathfrak{a}] \lhd \mathfrak{g}$, since $[\mathfrak{a},\mathfrak{a}] \blacktriangleleft \mathfrak{a}$ and $\mathfrak{a} \lhd \mathfrak{g}$. So by minimality either $\mathfrak{a}' = 0$ or $\mathfrak{a}' = \mathfrak{a}$. In the first case $\mathfrak{a}$ is an abelian ideal, hence solvable. But we just proved that the solvable radical is zero if the Killing form is non-degenerate. So $\mathfrak{a}' = \mathfrak{a}$.

Then $\mathfrak{a}^\perp$ and $\mathfrak{a} \cap \mathfrak{a}^\perp$ are also ideals. Again by minimality of $\mathfrak{a}$ there are two possibilities.

Suppose $\mathfrak{a} \cap \mathfrak{a}^\perp = \mathfrak{a}$. Let $Z \in \mathfrak{g}$ and $A \in \mathfrak{a}$ be arbitrary. There are $A_i, A_i' \in \mathfrak{a}$ such that $A = \sum_i [A_i, A_i']$, since $\mathfrak{a}' = \mathfrak{a}$. Then

$$B(A,Z) = \sum_i B([A_i, A_i'], Z) = \sum_i B(A_i, [A_i', Z]) = 0,$$

since $[A_i', Z] \in A$ and $A \subset A^\perp$. So $A \in \operatorname{rad} B = 0$. Contradiction.

So $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$. If $X \in \mathfrak{a}$ and $Y \in \mathfrak{a}^\perp$ then $[X,Y] \in \mathfrak{a} \cap \mathfrak{a}^\perp = 0$. So $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ is a direct sum of ideals. Since $B(X,Y) = 0$ if $X \in \mathfrak{a}$ and $Y \in \mathfrak{a}^\perp$, the Killing form on $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are non-degenerate. And we can use induction to complete the proof. Hence (i) implies (iii).

For the two converses we shall use Cartan's criterion.

Suppose (iii) holds, i.e. $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ a direct sum of ideals. We want to show that the Killing form is non-degenerate. The Killing form on $\mathfrak{g}$ is non-degenerate if and only if the Killing form on each $\mathfrak{g}_i$ is non-degenerate. So it suffices to prove that the Killing form of a simple Lie algebra is non-degenerate. So suppose $\mathfrak{g}$ is simple. The radical of the Killing form is an ideal, hence either $\mathfrak{g}$ or 0. In the first case it follows from Cartan's criterion that $\mathfrak{g}$ is solvable. Contradiction. Hence (iii) implies (i).

If the Killing form is degenerate, its radical is a non-trivial solvable ideal, so the radical of $\mathfrak{g}$ is non-zero, by Cartan's criterion, and $\mathfrak{g}$ is not semisimple. So (ii) implies (i). □

1.7. **Proof of Cartan's solvability criterion.** We still have to show that if $\mathrm{rad}(\mathfrak{g}) \subseteq \mathrm{rad}(B)$ then $\mathfrak{g}$ is solvable.

We shall use some facts from undergraduate algebra.

Theorem 1.5 (Chinese remainder theorem)**.** *Let k be a field and $f_1, \ldots, f_s \in k[T]$ be non-constant polynomials that are pairwise relatively prime. Then for any polynomials $g_1, \ldots, g_s$ there exists a polynomial g such that for all $1 \leq i \leq s$ we have*

$$g \equiv g_i \mod (f_i),$$

i.e., there exist polynomials h_i such that $g = g + f_i h_i$. This polynomial is unique modulo $(f_1 f_2 \ldots f_s)$.

The second result is called Lagrange interpolation.

Proposition 1.2. *Let k be a field and $a_0, a_1, a_2, \ldots, a_n$ be $n+1$ different elements of k. Then for any sequence $b_0, b_1, \ldots, b_n$ of elements in k, there exists a unique polynomial $F(T) \in k[T]$ of degree n such that*

$$F(a_i) = b_i,$$

for $0 \leq i \leq n$.

Proof. We are looking for a polynomial $F(T) = x_0 + x_1 T + x_2 T^2 + \ldots + x_n T n$ with unknown coefficients in k such that $F(a_i) = b_i$. So we have to solve the system of linear equations

$$1 \cdot x_0 + a_i \cdot x_1 + a_i^2 \cdot x_2 + \ldots a_i^n x_n = b_n.$$

The corresponding matrix $M \in \mathrm{Mat}(n+1 \times n+1, k)$ is the Vandermonde matrix

$$M_{ij} = a_i^j,$$

with $0 \leq i \leq n$ and $0 \leq j \leq n$. An exercise in linear algebra shows that its determinant is $\prod_{i<j}(a_i - a_j) \neq 0$. So we can solve uniquely the coefficients $x_0, x_1, \ldots, x_n$ of the polynomial.

For example with $n = 2$ we have to solve

$$\begin{pmatrix} 1 & a_0 & a_0^2 \\ 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}.$$

and we can if a_0, a_1, a_2 are all different. □

The third result follows from the existence of the Jordan normal form of a matrix.

Theorem 1.6 (Jordan normal form)**.** *Suppose V is an n-dimensional vector space over a field K and $\eta \in \mathrm{End}_K(V)$ a linear endomorphism such that all the eigenvalues lie in K (this is the certainly the case if $K = \mathbb{C}$).*

Let $\lambda_1, \ldots, \lambda_s$ be the various different eigenvalues of η. Define for each $1 \leq i \leq s$ the generalized eigenspace

$$V_{\lambda_i} := \{v \in V; (\eta - \lambda_i \mathbf{1}_V)^n(v) = 0.\}$$

If $v \in V_\lambda$ then $\eta(v) \in V_\lambda$ and

$$V = \oplus_{i=1}^s V_{\lambda_i}.$$

Let $\sigma \in \mathrm{End}_K(V)$ be uniquely defined by $\sigma(v) = \lambda_i v$, if $v \in V_{\lambda_i}$. In particular, there exists a basis of eigenvectors for the linear map σ. There exists a polynomial $P(T) \in K[T]$, without constant term, such that $P(\eta) = \sigma$. The difference $\nu = \eta - \sigma$ is nilpotent and $[\nu, \sigma] = 0$.

If $\eta = \sigma' + \nu'$ is another decomposition, such that there exists a basis of eigenvectors for σ', η' is nilpotent and $[\sigma', \nu'] = 0$, then $\sigma' = \sigma$ and $\eta' = \eta$.

The decomposition $\eta = \sigma + \nu$ is called the Jordan decomposition.

Proof. We do not need the full force of the Jordan normal form to prove this, nor the theory of invariant factors et cetera. Therefore we think it is still useful to give a proof here.

For any $\lambda \in K$ and $i \geq 0$ define

$$V_{\lambda,i} = \{v \in V; (\eta - \lambda\mathbf{1})^i = 0.\}$$

In particular $V_{\lambda,1}$ is the classical eigenspace of η with eigenvalue λ. We get a sequence of linear subspaces

$$0 = V_{\lambda,0} \subseteq V_{\lambda,1} \subseteq V_{\lambda,2} \subseteq \dots$$

If for some $i \geq 0$ we have $V_{\lambda,i} = V_{\lambda,i+1}$ then we claim that for all $j \geq i+1$ we also have $V_{\lambda,i} = V_{\lambda,j}$. Indeed, let $v \in V_{\lambda,j}$, i.e.

$$0 = (\eta - \lambda\mathbf{1})^j(v) = (\eta - \lambda\mathbf{1})^{i+1}((\eta - \lambda\mathbf{1})^{j-i-1}(v))$$

and so $(\eta - \lambda\mathbf{1})^{j-i-1}(v) \in V_{\lambda,i+1} = V_{\lambda,i}$ and so

$$(\eta - \lambda\mathbf{1})^{j-1}(v) = (\eta - \lambda\mathbf{1})^i((\eta - \lambda\mathbf{1})^{j-i-1}(v)) = 0$$

and $v \in V_{\lambda,j-1}$. It follows that if $V_{\lambda,j-1} \neq V_{\lambda,j}$ then $\dim V_{\lambda,j} \geq j$. We conclude that since $\dim V = n$ we must have that

$$V_{\lambda,n} = \{v \in V;\ \exists r \geq 0\ : (\eta - \lambda\mathbf{1})^r(v) = 0\}.$$

We also conclude that $V_{\lambda,j} = 0$ for all j if λ is not an eigenvalue for η.

Let $v \in V_{\lambda,n}$ then also $(\eta - \lambda\mathbf{1})^{n+1} = 0$ and so $\eta(v) - \lambda v \in V_{\lambda,n}$ and so $\eta(v) \in V_{\lambda,n}$.

Let $F(T) = \det(T\mathbf{1} - \eta) = \prod_{i=1}^s (T - \lambda_i)^{m_i}$ be the characteristic polynomial of η. Then by the Cayley-Hamilton theorem $F(\eta)$ acts trivially on V. By the Euclidian algorithm we can find $f(T), g(T) \in K[T]$ such that $f(t)\prod_{i=2}^s (T - \lambda_i)^{m_i} + g(T)(T - \lambda_1)^{m_1} = 1$. Define the linear endomorphisms π_1 and π_2 by

$$\pi_1(v) = f(\eta)\prod_{i=2}^{s}(\eta - \lambda_i\mathbf{1})^{m_i}(v)$$

and

$$\pi_2(v) = g(\eta)(\eta - \lambda_1\mathbf{1})^{m_1}(v).$$

Then $v = \pi_1(v) + \pi_2(v)$, $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0$, $\eta(\pi_1(v)) = \pi_1(\eta(v))$, $\eta(\pi_1(v)) = \pi_1(\eta(v))$ and $\pi_1(v) \in V_{\lambda_1}$. If $v \in V_{\lambda_1}$ then $\pi_1(v) = v$ and $\pi_2(v) = 0$ so $\pi_1^2 = \pi_1$ and $\pi_2^2 = \pi_2$. Put $U_1 = \pi_2(V) = \mathrm{Ker}\,\pi_1$. Then $V = V_{\lambda_1} \oplus U_1$ and both V_{λ_1} and U_1 are stable under η. The characteristic polynomial of η acting on U_1 is $\prod_{i=2}^s (T - \lambda_i)^{m_i}$ and its characteristic polynomial acting on V_{λ_1} is $(T - \lambda_1)$. We can then use induction to show that

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_s}.$$

By the Chinese remainder theorem there exists a polynomial $P(T) \in K[T]$ such that $P(T) \equiv \lambda_i \mod (T-\lambda_i)^n$, for $1 \leq i \leq s$ and additionally $P(T) \equiv 0 \mod T$, if all eigenvalues are non-zero. In any case, $P(T)$ has no constant term and for any i there exists a polynomial $h_i(T)$ such that $P(T) = \lambda_i + h_i(T)(T-\lambda_i)^n$.

We define $\sigma = P(\eta)$. Let $v \in V_{\lambda_i}$ then

$$\sigma(v) = P(\eta)(v) = \lambda_i v + h_i(\sigma)(\sigma - \lambda \mathbf{1})^n(v) = \lambda_i v.$$

So any non-zero vector of V_{λ_i} is an eigen-vector for σ with eigen-value λ_i. And $\sigma(\eta(v)) = P(\eta)(\eta(v)) = \eta(P(\eta)(v)) = \eta(\sigma(v))$ hence $[\sigma, \eta] = 0$. Define $\nu = \eta - \sigma$ then for any $v \in V_{\lambda_i}$ we have

$$\nu^n(v_i) = (\eta-\sigma)^n(v) = (\eta-\sigma)^{n-1}(\eta(v) - \lambda_i v) = (\eta-\sigma)^{n-1}(\eta - \lambda_i \mathbf{1})(v) = \ldots = (\eta - \lambda_i\mathbf{1})^n(v) = 0$$

(by induction). So $\nu^n = 0$ and so ν acts nilpotently and $[\sigma, \nu] = [\sigma, \eta - \sigma] = 0$.

Let $\eta = \sigma' + \eta'$ be another decomposition as in the theorem. Then σ' and ν' also commutes with η and hence with $P(\eta) = \sigma$ and with ν.

Now let $(\nu')^m = \nu^m = 0$ then $(\nu' - \nu)^{2m} = \sum_{i=0}^{2m} \binom{2m}{i} (\nu')^i(-\nu)^{2m-i} = 0$ since ν and ν' commute and for $0 \leq i \leq m$ either $(\nu')^i = 0$ or $(-\nu)^{2m-i} = 0$. So $\nu' - \nu = \sigma - \sigma'$ is nilpotent.

Let $v \in V_{\lambda_i}$ then $\sigma(\sigma'(v)) = \sigma'(\sigma(v)) = \lambda_i(\sigma'(v)$ so $\sigma'(v) \in V_{\lambda_i}$ and so σ' preserves every V_{λ_i}. Let $v \in V$ be any eigenvector for σ' with eigenvalue μ. We can write uniquely $v = v_1 + v_2 + \ldots + v_s$, where $v_i \in V_{\lambda_i}$ and since $\sigma'(v) = \mu v$ we get $\sum_{i=1}^s \mu v_i = \sum_{i=1}^s \sigma(v_i)$. Since $\sigma(v_i) \in V_{\lambda_i}$ we get $\sigma'(v_i) = \mu v_i$ for each i. We get $(\sigma - \sigma')(v_i) = (\lambda_i - \mu)v_i$ and so if $v_i \neq 0$ then $(\lambda_i - \mu)$ is an eigenvalue of $\sigma - \sigma' = \nu' - \nu$ which is nilpotent (who has only 0 as eigenvalue). So $\lambda_i = \mu$ and there is a unique i such that $v_i \neq 0$. We conclude that any eigenvector for σ' is necessarily also an eigenvector for σ. So $\sigma' - \sigma$ also has a basis of eigenvectors. Since 0 is the only eigenvalue we conclude that $\sigma' - \sigma = 0$. This finishes the proof. □

Lemma 1.6. *Let V be a vector space over a field, $\eta \in \mathrm{End}_K(V)$ such that all its eigenvalues are in K. Let $\eta = \sigma + \nu$ be its Jordan decomposition. Let* ad *be the adjoint representation of the Lie algebra $\mathrm{End}_K(V)$. Then $\mathrm{ad}(\eta) = \mathrm{ad}(\sigma) + \mathrm{ad}(\nu)$ is the Jordan decomposition of $\mathrm{ad}(\eta) \in \mathrm{End}_K(\mathrm{End}_K(V))$. If $\lambda_1, \ldots, \lambda_s$ are the eigenvalues of η, then $\{\lambda_i - \lambda_j, i \neq j\}$ are the eigenvalues of $\mathrm{ad}(\eta)$.*

Proof. We pick bases in each V_{λ_i}, for each eigenvalue λ_i; and the reunion of those basis gives a basis $e_1, \ldots, e_n$ of V. There are eigenvalues μ_i such that $\sigma e_i = \mu_i e_i$. Let $x_1, \ldots, x_n \in V^*$ be the dual basis. Then a basis for $\mathrm{End}_K(V)$ is E_{ij} defined by $E_{ij}(v) = x_j(v)e_i$. Then for $v = \sum_r c_r e_r$:

$$\begin{aligned}
[\mathrm{ad}(\sigma)(E_{ij})](v) &= \sum_r c_r(\sigma(E_{ij}(e_r)) - E_{ij}(\sigma(e_r))) \\
&= \sum_r c_r(\sigma(x_j(e_r)e_i) - E_{ij}(\mu_r e_r)) \\
&= \sum_r c_r(x_j(e_r)\mu_i e_i - (\mu_r x_j(e_r)e_i) \\
&= c_j(\mu_i - \mu_j)e_i \\
&= (\mu_i - \mu_j)x_j(v)e_i \\
&= (\mu_i - \mu_j)E_(ij)(v)
\end{aligned}$$

so $\mathrm{ad}(\sigma)(E_{ij}) = (\mu_i - \mu_j)E_(ij)$ and the E_{ij} form a basis of eigenvectors for $\mathrm{ad}(\sigma)$.

We saw earlier that if ν is nilpotent, then $\mathrm{ad}(\nu)$ is also nilpotent. Finally $[\mathrm{ad}(\sigma), \mathrm{ad}(\nu)] = \mathrm{ad}([\sigma, \nu]) = 0$. So indeed $\mathrm{ad}(\eta) = \mathrm{ad}(\sigma) + \mathrm{ad}(nu)$ is the Jordan decomposition of $\mathrm{ad}(\eta)$. □

Now we are ready to prove Cartan's solvability criterion.

Proof of Cartan's solvability criterion. Assuming that $[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{rad}(B)$ it remains to be shown that $\mathfrak{g}$ is solvable. We shall show that $\mathrm{ad}(Y)$ is nilpotent for any $Y \in [\mathfrak{g}, \mathfrak{g}]$, then from Engel's theorem it will follow that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent and hence $\mathfrak{g}$ will be solvable.

We shall only prove the case where $k \subset \mathbb{C}$. It can be proved in general too by modifying the arguments somewhat. Let $V = \mathfrak{g} \otimes_k \mathbb{C}$ be the complex vector space associated to $\mathfrak{g}$. In particular, any k-basis of $\mathfrak{g}$ becomes a $\mathbb{C}$-basis of $\mathrm{End}_{\mathbb{C}}(V)$. On $\mathrm{End}_{\mathbb{C}} V$ we have the $\mathbb{C}$-bilinear form $C(X, Y) = \mathrm{tr}(XY) \in \mathbb{C}$. By replacing $\mathfrak{g}$ by $\mathrm{ad}(\mathfrak{g})$ we may suppose that $\mathfrak{g} \subset \mathrm{End}_{\mathbb{C}} V$ is a Lie sub-algebra over k, and $[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{rad} C$. The hypothesis becomes then that if $Y_1, Y_2, Y_3 \in \mathfrak{g}$ then $C([Y_1, Y_2], Y_3) = 0$

Let $Y \in [\mathfrak{g}, \mathfrak{g}]$. We must prove that it is nilpotent. Let $\lambda_1, \ldots, \lambda_s$ be its eigenvalues acting on V and the generalized eigenspace decomposition $V = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_s}$. There are $S, N \in \mathrm{End}_C(V)$ such that $Y = S + N$, $[S, N] = 0$, N is nilpotent and S acts by on V_{λ_i} by multiplication by λ_i. Define $\overline{S}$ by $\overline{S}v = \overline{\lambda_i}v$ if $v \in V_{\lambda_i}$. Since Y and S preserve each V_{λ_i} and $\overline{S}$ acts by a scalar on V_{λ_i} it follows that $\overline{S}$ also commutes with N and so $\overline{S}N$ is nilpotent.

We will calculate $C(\overline{S}, Y) = \mathrm{tr}(\overline{S}Y) = \mathrm{tr}(\overline{S}S) + \mathrm{tr}(\overline{S}N) = \mathrm{tr}(\overline{S}S)$ in two different ways. Restricted to V_{λ_i} its trace becomes $\overline{\lambda_i}\lambda_i \dim_C(V_{\lambda_i}) \geq 0$ and $= 0$ if 0 is the only eigenvalue, hence if Y is nilpotent. This is what we want to prove.

The Lie algebra $\mathrm{End}_{\mathbb{C}}(V)$ also has an adjoint representation, call it ad_0. And by the previous lemma $\mathrm{ad}_0(Y) = \mathrm{ad}_0(S) + \mathrm{ad}_0(N)$ is the Jordan decomposition of $\mathrm{ad}_0(Y)$. So there is a polynomial without constant term $P(T) \in \mathbb{C}[T]$ such that $P(\mathrm{ad}_0(Y)) = \mathrm{ad}_0(S)$.

By Lagrange interpolation, there is a polynomial $Q(T) \in \mathbb{C}[T]$ such that $Q(\lambda_i - \lambda_j) = \overline{\lambda_i} - \overline{\lambda_j}$. Then $Q(\mathrm{ad}_0(S)) = \mathrm{ad}_0(\overline{S})$. By composition we get a polynomial $R(T) = r_1 T + r_2 T^2 + \ldots + mT^m$, with $r_i \in \mathbb{C}.R(T)$, such that $R(\mathrm{ad}_0(Y)) = \mathrm{ad}(\overline{S})$ and $R(0) = 0$. Or $\mathrm{ad}(\overline{S}) = \sum_{j=1}^m r_j \,\mathrm{ad}_0(Y)^j$.

Since $Y \in [\mathfrak{g}, \mathfrak{g}]$ there exist $X_i, Z_i \in \mathfrak{g}$ such that $Y = \sum_i [X_i, Z_i]$.

$$\begin{aligned}
C(\overline{S}, Y) &= \sum_i C(\overline{S}, [X_i, Z_i]) \\
&= \sum_i C([\overline{S}, X_i], Z_i) \\
&= \sum_i C(\mathrm{ad}_0(\overline{S})(X_i), Z_i) \\
&= \sum_i \sum_{j=1}^m r_j C(\mathrm{ad}_0(Y)^j(X_i), Z_i) \\
&= \sum_i \sum_{j=1}^m r_j C([Y, \mathrm{ad}_0(Y)^{j-1}(X_i)], Z_i) \\
&= 0
\end{aligned}$$

since for any $j \geq 1$ we have $Y, \mathrm{ad}_0(Y)^{j-1}(X_i), Z_i \in \mathfrak{g}$ and so by hypothesis $C([Y, \mathrm{ad}_0(Y)^{j-1}(X_i)], Z_i) = 0$. (Remark: our hypothesis cannot be used directly to show that $C([\overline{S}, X_i], Z_i) = 0$, since we have no reason to believe that $\overline{S} \in \mathfrak{g}$.)

So indeed Y is nilpotent. This finishes the proof. □

Having done so much work, it is pleasant to be able to use the same proof for another result. We call a representation $\rho : \mathfrak{g} \to \mathrm{End}_k V$ *faithful* if ρ is injective; or equivalently if $X \cdot v = 0$ for all $v \in V$ then $X = 0$.

Proposition 1.3. *Let V be a faithful representation for the Lie algebra $\mathfrak{g}$. And suppose that $[\mathfrak{g}, \mathfrak{g}]$ is contained in the radical of $B_V(\cdot, \cdot)$. Then $\mathfrak{g}$ is solvable.*

Proof. We shall assume $k \subseteq \mathbb{C}$, although the proof can be adjusted for the more general case. Replacing V by its complexification $V^{\mathbb{C}}$, and by defining the form $C(X, Y) = \mathrm{tr}(X, Y)$ on $\mathrm{End}_C(V^{\mathbb{C}})$ we end up in the same situation as we were after the second paragraph of the proof of Cartan's solvability criterion. So we can use the same proof to conclude that every element of $[\mathfrak{g}, \mathfrak{g}]$ is acting nilpotently on V, so by Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, and $\mathfrak{g}$ is solvable. □

We get a corollary generalizing the implication (ii)⇒(i) of Cartan's theorem.

Corollary 1.5. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and V a faithful $\mathfrak{g}$-representation. Then the form $B_V(\cdot, \cdot)$ is non-degenerate.*

Proof. The radical of $B_V(\cdot, \cdot)$ is a Lie-algebra acting faithfully on V, so the hypothesis of the proposition is satisfied. Hence this radical is a solvable ideal of $\mathfrak{g}$ but $\mathfrak{g}$ has no non-zero solvable ideals. □

2. Representation theory for $\mathfrak{sl}_2(k)$

We shall now give a classification of the irreducible (finite dimensional) representations for the simple Lie algebra $\mathfrak{sl}_2(k)$. We will not need to assume that k is algebraically closed (but we still assume k to be of characteristic 0). (There are also simple infinite dimensional representations.)

Consider the polynomial ring $k[X_0, X_1]$. Partial derivation $\partial_0 := \frac{\partial}{\partial X_0}$ and $\partial_1 := \frac{\partial}{\partial X_1}$ are defined as usual. They are k-linear operators on $k[X_0, X_1]$ satisfying Leibnitz' rule, for example

$$\partial_0(PQ) = \partial_0(P)Q + P\partial_0(Q),$$

for polynomials P and Q. These operators commute $[\partial_0, \partial_1] = 0$.

For any polynomials f_0, f_1 the operator $f_0\partial_0 + f_1\partial_1$ also acts on $k[X_0, X_1]$ by

$$(f_0\partial_0 + f_1\partial_1)(P) := f_0\partial_0(P) + f_1\partial_1(P);$$

it is also k-linear and satisfies Leibnitz' rule. Two linear operators of this kind do no longer commute in general, but their Lie bracket is of the same form. We calculate this Lie bracket of $D_1 = (f_0\partial_0 + f_1\partial_1)$ and $D_2 = (g_0\partial_0 + g_1\partial_1)$:

$$\begin{aligned}
[D_1, D_2](P) &= [(f_0\partial_0 + f_1\partial_1), (g_0\partial_0 + g_1\partial_1)](P) \\
&= (f_0\partial_0 + f_1\partial_1)\left((g_0\partial_0 + g_1\partial_1)(P)\right) - (g_0\partial_0 + g_1\partial_1)\left((f_0\partial_0 + f_1\partial_1)(P)\right) \\
&= (f_0\partial_0 + f_1\partial_1)\left((g_0\partial_0(P) + g_1\partial_1(P)\right) - (g_0\partial_0 + g_1\partial_1)\left((f_0\partial_0(P) + f_1\partial_1(P)\right) \\
&= f_0\partial_0(g_0\partial_0(P)) + f_0\partial_0(g_1\partial_1(P)) + f_1\partial_1(g_0\partial_0(P)) + f_1\partial_1(g_1\partial_1(P) \\
&\quad - (g_0\partial_0(f_0\partial_0(P)) + g_0\partial_0(f_1\partial_1(P)) + g_1\partial_1(f_0\partial_0(P)) + g_1\partial_1(f_1\partial_1(P))) \\
&= f_0\partial_0(g_0)\partial_0(P) + f_0g_0\partial_0(\partial_0(P)) + f_0\partial_0(g_1)\partial_1(P) + f_0g_1\partial_0(\partial_1(P)) \\
&\quad + f_1\partial_1(g_0)\partial_0(P) + f_1g_0\partial_1(\partial_0(P)) + f_1\partial_1(g_1)\partial_1(P) + f_1g_1\partial_1(\partial_1(P)) \\
&\quad - g_0\partial_0(f_0)\partial_0(P) - g_0f_0\partial_0(\partial_0(P)) - g_0\partial_0(f_1)\partial_1(P) - g_0f_1\partial_0(\partial_1(P)) \\
&\quad - g_1\partial_1(f_0)\partial_0(P) - g_1f_0\partial_1(\partial_0(P)) - g_1\partial_1(f_1)\partial_1(P) - g_1f_1\partial_1(\partial_1(P)) \\
&= \left(f_0\partial_0(g_0) + f_1\partial_1(g_0) - g_0\partial_0(f_0) - g_1\partial_1(f_0)\right)\partial_0(P) \\
&\quad + \left(f_0\partial_0(g_1) + f_1\partial_1(g_1) - g_0\partial_0(f_1) - g_1\partial_1(f_1)\right)\partial_1(P) \\
&= \left((f_0\partial_0 + f_1\partial_1)(g_0) - (g_0\partial_0 + g_1\partial_1)(f_0)\right)\partial_0(P) \\
&\quad + \left((f_0\partial_0 + f_1\partial_1)(g_1) - (g_0\partial_0 + g_1\partial_1)(f_1)\right)\partial_1(P) \\
&= \left(D_1(g_0) - D_2(f_0)\right)\partial_0(P) + \left(D_1(g_1) - D_2(f_1)\right)\partial_1(P)
\end{aligned}$$

We conclude that

$$[D_1, D_2] = \left(D_1(g_0) - D_2(f_0)\right)\partial_0 + \left(D_1(g_1) - D_2(f_1)\right)\partial_1$$

is indeed of the same form. The linear space of these linear operators on $k[X_0, X_1]$

$$L := \{f_0\partial_0 + f_1\partial_1; f_0, f_1 \in k[X_0, X_1]\}$$

becomes an infinite dimensional Lie algebra over k with infinite dimensional k-representation on $k[X_0, X_1]$.

It has a simple Lie sub-algebra of dimension three generated by

$$E := X_0\partial_1, F := X_1\partial_0, H := X_0\partial_0 - X_1\partial_1$$

with brackets

$$[E, F] = H, [H, E] = 2E, [H, F] = -2F.$$

It follows already that it equals its derived Lie-algebra.

We calculate

$$E(X_0) = 0, E(X_1) = X_0, F(X_0) = X_1, F(X_1) = 0, H(X_0) = X_0, , H(X_1) = -X_1$$

so $V_1 := kX_0 \oplus kX_1$ is a sub-representation; and on the basis X_0, X_1 the generators take the form

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, aE + bF + cH = \begin{pmatrix} c & a \\ b & -c \end{pmatrix},$$

and so we have an isomorphism

$$kE + kF + kH = \mathfrak{sl}(V_1) \simeq \mathfrak{sl}_2(k).$$

We observe $E(X_0^i) = 0$ and $E(X_0^i X_1^j) = jX_0^{i+1}X_1^{j-1}$ if $j \geq 0$, $F(X_1^j) = 0$ and $F(X_0^i X_1^j) = iX_0^{i-1}X_1^{j+1}$ if $j \geq 0$ and $H(X_0^i X_1^j) = (i-j)X_0^i X_1^j$. So the collection V_d of all polynomials of degree d is a sub-representation of $k[X_0, X_1]$ for our Lie algebra $kE + kF + kH$, with basis $X_0^d, X_0^{d-1}X_1, X_0^{d-2}X_1^2, \ldots, X_1^d$.

In fact, we prefer a slightly different basis. Put $v_0 = X_0^d$, and by induction $v_{i+1} = F(v_i)$. Explicitly,

$$v_0 = X_0^d, v_1 = dX_0^{d-1}X_1, v_2 = d(d-1)X_0^{d-2}X_1^2, \ldots, v_i = \frac{d!}{(d-i)!}X_0^{d-i}X_1^i, \ldots, v_d = d!X_1^d, v_{d+1} = 0.$$

And

$$E(v_i) = X_0\partial_1(\frac{d!}{(d-i)!}X_0^{d-i}X_1^i) = \frac{d!}{(d-i)!}(i)X_0^{d-i+1}X_1^{i-1} = i(d-i+1)v_{i-1}.$$

And

$$E^i(v_i) = \frac{i!d!}{(d-i)!}v_0 \text{ and so } F^iE^iv_i = \frac{i!d!}{(d-i)!}v_i.$$

Since every polynomial is a unique sum of homogeneous polynomials of various degrees, we get a direct sum $k[X_0, X_1] = \oplus_{i=0}^{\infty} V_i$ of $kE + kF + kH$-representations.

We calculate the bilinear form associated to V_1 using this basis X_0, X_1:

$$B_{V_1}\left(aE + bF + cH, a'E + b'F + c'H\right) = \text{tr}\left(\begin{pmatrix} c & a \\ b & -c \end{pmatrix} \cdot \begin{pmatrix} c' & a' \\ b' & -c' \end{pmatrix}\right) = ab' + ba' + 2cc'.$$

The dual bases with respect to this form becomes $E' = F, F' = E, H' = \frac{1}{2}H$. Form the quadratic differential operator, called the *Casimir operator*

$$\Delta = EE' + FF' + HH' = EF + FE + \frac{1}{2}HH.$$

It still acts linearly on $k[X_0, X_1]$, but the Liebniz' rule does no longer hold. In fact it acts on any $\mathfrak{sl}_2(k)$-representation. Let V be any $kE + kE + kH$ representation. Then we define

$$\Delta \cdot v := E \cdot F \cdot v + F \cdot E \cdot v + \frac{1}{2} H \cdot H \cdot v$$

for $v \in V$. It commutes with the action of $kE + kE + kH = \mathfrak{sl}(V_1) \simeq \mathfrak{sl}_2(k)$. For example

$$\begin{aligned}
[\Delta, E] &= EFE + FEE + \frac{1}{2}HHE - EEF - EFE - \frac{1}{2}EHH \\
&= ([F,E]E + EFE) + \frac{1}{2}(H[H,E] + HEH) - (E[E,F] + EFE) - \frac{1}{2}([E,H]H + HEH) \\
&= -HE + \frac{1}{2}(2HE) - EH - \frac{1}{2}(-2EH) \\
&= 0.
\end{aligned}$$

How does Δ act on V_d? We calculate on monomials.

$$\begin{aligned}
\Delta(X_0^i X_1^j) &= (EE' + FF' + HH')(X_0^i X_1^j) = (EF + FE + \frac{1}{2}HH)(X_0^i X_1^j) \\
&= E(iX_0^{i-1}X_1^{j+1}) + F(jX_0^{i+1}X_1^{j-1}) + \frac{1}{2}H((i-j)X_0^i X_1^j) \\
&= (i(j+1) + (i+1)j + \frac{1}{2}(i-j)^2)(X_0^i X_1^j) \\
&= ((i+j) + \frac{1}{2}(i+j)^2)(X_0^i X_1^j).
\end{aligned}$$

So Δ acts on V_d by scalar multiplication with $\frac{d(d+2)}{2}$. The Casimir operator can be defined for any simple Lie algebra; it acts on any representation, commuting with the Lie algebra action, by

$$\Delta \cdot v = E \cdot F \cdot v + F \cdot E \cdot v + \frac{1}{2} H \cdot H \cdot v.$$

So if V is any $\mathfrak{sl}_2(k)$-representation, the Casimir operator gives rise to an endomorphism

$$\Delta : V \to V$$

of $\mathfrak{sl}_2(k)$-representations.

The *Euler operator* $\eta = X_0\partial_0 + X_1\partial_1$ acting on $k[X_0, X_1]$ also commutes with $kE + kF + kH$. A polynomial $P \in k[X_0, X_1]$ is homogeneous of degree d if and only if $\eta(P) = dP$ (it suffices to check this on monomials, and then it is easy). But the Euler-operator is not as useful in Lie-theory as the Casimir operator, since it does not act naturally on all $\mathfrak{sl}_2(k)$-representations and it does not generalize to all simple Lie algebras. In fact, $kE + kF + kH + k\eta$ is isomorphic to $\mathfrak{gl}(V_1)$ with η corresponding to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, when we choose the basis X_0, X_1 for V_1.

Lemma 2.1. *Each representation V_d is a simple $\mathfrak{sl}_2(k)$-module.*

Proof. Let $U \subseteq V_d$ be an $\mathfrak{sl}_2(k)$ subrepresentation, and $u = \sum_{i=0}^{d} c_i X_0^i X_1^{d-i} \in U$ be any non-zero element. Let r be maximal, such that $c_r \neq 0$. Then $E^{d-r} \cdot u = E \cdot E \cdot E \cdot \ldots \cdot E \cdot u = (d-r)! c_r X_0^d$ is also in U. Now applying F-several times, shows that each $X_0^i X_1^{d-i} \in U$. Hence $U = V_d$. □

Theorem 2.1. *Let V be a simple $\mathfrak{sl}_2(k)$-representation of dimension $d+1$.*

(i) There is an isomorphism $V_d \simeq V$ of representations.

(ii) Let $\phi : V_d \to V_d$ be an endomorphism as $\mathfrak{sl}_2(k)$-representations. Then there is a unique $c \in k$ such that $\phi = c\mathbf{1}_{V_d}$

Proof. (i) Let $\lambda \in K \supset k$ be an eigenvalue for H acting on V, in some extension field of k. We want to show that $\lambda \in \mathbb{Z}$, and we can therefore take $k = K$ after all. Hence there is a non-zero $v \in V \otimes_k K$ such that $H \cdot v = \lambda v$. Then the following calculation

$$H \cdot (E \cdot v) = [H, E] \cdot v + E \cdot H \cdot v = 2E \cdot v + \lambda = (\lambda + 2)(E \cdot v)$$

shows $E \cdot v$ is also an eigenvector for H of eigenvalue $\lambda + 2$, if it is non-zero. Without loss of generality we can assume that $E \cdot v = 0$. Then

$$H \cdot (F \cdot v) = [H, F] \cdot v + F \cdot H \cdot v = -2F \cdot v + \lambda = (\lambda - 2)(F \cdot v)$$

shows that if $F \cdot v \neq 0$ then it is also an eigenvector for H with eigenvalue $\lambda - 2$. Put $v_0 = v$ and inductively , $v_{i+1} = F \cdot v_i$. Let s be minimal such that $v_{s+1} = 0$. Then $H \cdot v_i = (\lambda - 2i)v_i$ and so $v_0, v_1, \ldots, v_s$ are linearly independent, since they are eigenvectors with different eigenvalues. By induction on for $i \geq 1$ we prove that

$$E \cdot v_i = i(\lambda - i + 1)v_{i-1}.$$

For $i = 1$:

$$E \cdot v_1 = E \cdot F \cdot v_0 = [E, F] \cdot v_0 + F \cdot E \cdot v_0 = H \cdot v_0 = \lambda v_0;$$

and supposing true for $i - 1$:

$$\begin{aligned} E \cdot v_i &= E \cdot F \cdot v_{i-1} = [E, F] \cdot v_{i-1} + F \cdot E \cdot v_{i-1} \\ &= H \cdot v_{i-1} + F \cdot (i-1)(\lambda - i + 2)v_{i-2} \\ &= ((\lambda - 2(i-1)) + (i-1)(\lambda - i + 2))v_{i-1} \\ &= i(\lambda - i + 1)v_{i-1} \end{aligned}$$

But since $v_{s+1} = 0$ we get $0 = E \cdot v_{s+1} = (s+1)(\lambda - s)v_s = 0$ hence $\lambda = s$.

So the eigenvalue λ is in fact an integer, and we could have taken $K = k$ and $U := kv_0 + \ldots + kv_s$ is a non-zero subrepresentation of the simple representation V. Hence we get equality, and in particular $s = d$.

Now we send X_0^d to v_0 and $F^i \cdot X_0^d = \frac{d!}{(d-i)!} X_0^{d-i} X_1^i$ to v_i to get the isomorphism, since the actions of E, F and H coincident.

(ii) Since ϕ is an endomorphism of $\mathfrak{sl}_2(k)$-representations and the $F^i \cdot X_0^d$ form a basis of V_d, ϕ is completely determined by $\phi(X_0^d)$, since $\phi(F^i \cdot X_0^d) = F^i \cdot \phi(X_0^d)$. If $\phi(X_0^d) = 0$ then $\phi = 0$, otherwise $\phi(X_0^d)$ is an H-eigenvector with eigenvalue d, hence there exists a unique $c \in k$ such that $\phi(X_0^d) = cX_0^d$ and so $\phi(F^i \cdot X_0^d) = cF^i \cdot X_0^d$ and $\phi = c\mathbf{1}$. □

Example 2.1. Since $\mathfrak{sl}_2(k)$ is a simple Lie algebra, the adjoint representation is simple of dimension three and so isomorphic to V_2. The representation on V_d^* is also simple with dimension $d+1$, hence $V_d \simeq V_d^*$ as $\mathfrak{sl}_2(k)$-representations.

Corollary 2.1. *Let V be a finite dimensional $\mathfrak{sl}_2(k)$-representation.*

(i) All the eigenvalues of Δ acting on V are non-negative half-integers of the form $\frac{d(d+2)}{2}$ for $d \in \mathbb{Z}_{\geq 0}$.

(ii) $\mathrm{tr}_V(\Delta) = 0$ if and only if 0 is the only eigenvalue for Δ acting on V if and only if Δ acts as 0 on V if and only if $\mathfrak{sl}_2(k)$ acts trivially on V.

Proof. Proof by induction on $\dim V$. If it is one-dimensional, $V \simeq V_0$ is the trivial $\mathfrak{sl}_2(k)$-representation, with Δ acting by 0 and all statements become trivial.

If V is simple, it is isomorphic to V_d for some d, on which Δ acts by multiplication by the non-negative half-integer $\frac{d(d+2)}{2}$, which is 0 if and only if $d = 0$ if and only if $\mathfrak{sl}_2(k)$ acts trivially.

If V is not simple, let $U \simeq V_d$ be a simple sub-representation. Since Δ commutes with the action of $\mathfrak{sl}_2(k)$, it acts on both U and V/U. So by induction all eigenvalues of Δ are non-negative half-integers on U and V/U, hence also on V. This proves i.

Since $\mathrm{tr}_V(\Delta) = \mathrm{tr}_U(\Delta) + \mathrm{tr}_{V/U}(\Delta)$ and all eigenvalues for Δ are non-negative, it follows that $\mathrm{tr}_V(\Delta) = 0$ if and only if $\mathrm{tr}_U(\Delta) = 0$ and $\mathrm{tr}_{V/U}(\Delta) = 0$.

Suppose $\mathrm{tr}_V(\Delta) = 0$, hence $\mathrm{tr}_U(\Delta) = 0$ and $\mathrm{tr}_{V/U}(\Delta) = 0$. Then by induction Δ acts trivially on both U and V/U, so in particular 0 is the only eigenvalue of Δ acting on V, and both U and V/U are trivial representations for $\mathfrak{sl}_2(k)$.

Let $v \in V$, then $E \cdot \overline{v} = 0$, for $\overline{v} \in V/U$, so $F \cdot E \cdot v = E \cdot v \in U$ and $E \cdot E \cdot v = 0$. This is because $\mathfrak{sl}_2(k)$ acts trivially on U and V/U. Similarly $F \cdot v$ and $H \cdot v$ are all in U, and $F \cdot E \cdot v = E \cdot F \cdot v = H \cdot H \cdot v = 0$ and so $\Delta \cdot v = 0$. i.e., Δ acts trivially on V. Also $H \cdot v = [E, F] \cdot v = E \cdot F \cdot v - F \cdot E \cdot v = 0$, and similarly $E \cdot v = \frac{1}{2}[H, E] \cdot v = 0$ and $F \cdot v = -\frac{1}{2}[H, F] \cdot 0$, so indeed $\mathfrak{sl}_2(k)$ acts trivially on the whole of V.

Conversely, if $\mathfrak{sl}_2(k)$ acts trivially on V, then Δ acts as 0 and with trace 0. This finishes the proof. □

Lemma 2.2. *Let $U \subset V$ be a sub-representation for $\mathfrak{sl}_2(k)$ of codimension one. There is a $v \in V$ such that $X \cdot v = 0$ for all $X \in \mathfrak{sl}_2(k)$ and such that $V = U \oplus kv$ as $\mathfrak{sl}_2(k)$-representations.*

Proof. We know that all eigenvalues of Δ acting on V are non-negative integers. Let $\lambda_1, \ldots, \lambda_s$ be the different eigenvalues. For $\lambda \in k$ define

$$V_{\lambda,i} := \{v \in V; (\Delta - \lambda)^i \cdot v = 0\}$$

and $V_\lambda = V_{\lambda,n}$ the generalized eigenspace (here $n = \dim_k V$). Since Δ commutes with the $\mathfrak{sl}_2(k)$ action, each V_λ is an $\mathfrak{sl}_2(k)$-representation and we get a direct sum decomposition

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \ldots \oplus V_{\lambda_s}$$

of $\mathfrak{sl}_2(k)$-representations. Put $U_i = U \cap V_{\lambda_i}$, then $V/U \simeq \oplus_{i=1}^s V_{\lambda_i}/U_i$, and so there is a unique i such that $U_i \neq V_{\lambda_i}$. Without loss of generality we can assume that $i = 1$

Let j be maximal such that $V_{\lambda_1,j} \subset U_1$, then $U_1 \subset V_{\lambda_1,j+1}$ is a proper inclusion. Since $\dim(V_{\lambda_1}/U_1) = 1$ we get necessarily that $V_{\lambda_1} = V_{\lambda_1,j+1}$. Since λ_1 is the only eigenvalue for Δ acting on V_{λ_1} and U_1 we get

$$\lambda_1 \dim_k V_{\lambda_1} = \mathrm{tr}_{V_{\lambda_1}}(\Delta) = \mathrm{tr}_{U_1}(\Delta) + \mathrm{tr}_{V_{\lambda_1}/U_1}(\Delta) = \lambda_1 \dim_k U_1 + 0 = \lambda_1(\dim_k V_{\lambda_1} - 1)$$

since $\mathfrak{sl}_2(k)$ and Δ act trivially on any one-dimensional $\mathfrak{sl}_2(k)$-representation.

We conclude that $\lambda_1 = 0$, and we can apply the corollary before. So Δ and $\mathfrak{sl}_2(k)$ act trivially on V_{λ_1}. Now let $v \in V_{\lambda_1}$ such that $v \notin U_1$. Then $V_{\lambda_1} = U_1 \oplus kv$ as (trivial) $\mathfrak{sl}_2(k)$-representations and we get a direct sum decomposition as $\mathfrak{sl}_2(k)$ representations

$$V = (kv \oplus U_1) \oplus V_{\lambda_2} \oplus \ldots \oplus V_{\lambda_s} = kv \oplus U_1 \oplus U_2 \oplus \ldots \oplus U_s = kv \oplus U.$$

This finishes the proof. □

Theorem 2.2 (Reductivity of $\mathfrak{sl}_2(k)$-representations)**.** *Let V be any (finite dimensional) representation for $\mathfrak{sl}_2(k)$.*

(i) Let $U \subset V$ be a subrepresentation. Then there is a subrepresentation U' such that $V = U \oplus U'$ is a direct sum of representations.

(ii) There is a direct sum decomposition as $\mathfrak{sl}_2(k)$-representations

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_s$$

where each U_i is a simple $\mathfrak{sl}_2(k)$ representation isomorphic to V_d, when $\dim U_i = d + 1$.

The direct sum $V_{(d)}$ of all U_i with dimension $d+1$ is the same as the Δ-eigenspace with eigenvalue $\frac{d(d+1)}{2}$, so does not depend on the decomposition as a direct sum of simple sub-representations.

Proof. (i) The vector space $\mathrm{End}_k(V)$ is also an $\mathfrak{sl}_2(k)$-representation when we define $X \cdot \phi$ (for $X \in \mathfrak{sl}_2(k)$ and $\phi \in \mathrm{End}_k(V)$) by

$$[X \cdot \phi](v) = X \cdot \phi(v) - \phi(X \cdot v).$$

It has the subrepresentations

$$M := \{\phi \in \mathrm{End}\, V; \phi(V) \subset U; \exists \lambda \in k : \forall u \in U : \phi(u) = \lambda u\},$$

and

$$N := \{\phi \in \mathrm{End}\, V; \phi(V) \subset U; \forall u \in U : \phi(u) = 0\}.$$

Then $N \subset M$ and M/N is one-dimensional.

Using the previous lemma there exists a $\phi \in M$ such $k\phi$ is the trivial representation and $M = N \oplus k\phi$. By normalizing we can assume that ϕ restricted to U acts by the scalar 1. Which means for all $v \in V : \phi(v) \in U$ and for all $X \in \mathfrak{sl}_2(k)$ we have $X \cdot \phi(v) = \phi(X \cdot v)$ and $\phi(u) = u$ for all $u \in U$. Or ϕ is a morphism of representations. Take $U' = \mathrm{Ker}\, \phi$, then U' is a subrepresentation such that $U \cap U' = 0$. If $v \in V$ then $u := \phi(v) \in U$ and $\phi(\phi(v) - v) = \phi(v) - \phi(v) = 0$ so $u' := \phi(v) \in U'$ and $v = u + u'$ We proved $V = U \oplus U'$ as representations, and so (i) follows.

(ii) Start with any simple sub-representation U_1 with complement U_1' as in (i), then decompose U_1'. In this way we end u with a direct sum decomposition where each factor is simple and so isomorphic to some V_d. The remaining statement directly. □

To sum up.

Corollary 2.2. *Let V be any (finite dimensional) representation for $\mathfrak{sl}_2(k)$. Both E and F act nilpotently on V. There is a basis for V that consists of simultaneous eigenvectors for both H and the Casimir operator Δ. All the eigenvalues for E, F, H are integers, and for Δ non-negative half-integers.*

The space $V^{E=0} := \{v \in V;\ E \cdot v = 0\}$ is H-stable, and all its eigenvalues are non-negative integers. Let n_d be the dimension of the H-eigenspace on $V^{E=0}$ of eigenvalue d. Then n_d is the number of copies isomorphic to V_d in any direct sum decomposition of V of simple $\mathfrak{sl}_2(k)$-representations. And $\mathrm{tr}_V(\Delta) = \sum_{d\geq 0} \frac{n_d d(d+1)(d+2)}{2}$.

Proof. Since V is the driect sum of simple representations, we can assume that V is simple, hence isomorphic to one of the V_n. By the explicit construction, the properties were already checked. □

Corollary 2.3. *Let V be a any (finite dimensional) representation for $\mathfrak{sl}_2(k)$. The eigenvalues for H are all integers. Put n_i for the dimension of the H-eigenspace with eigenvalue i.*

(i) Then $n_i = n_{-i}$.

(ii) n_0 is the number of irreducible components of V isomorphic to V_d, with d even. And n_1 is the number of irreducible components of V isomorphic to V_d, with d odd. So $n_0 + n_1$ is the number of irreducible components.

Proof. If suffices to prove in the case of simple V. Then it is clear by the explicit knowledge of the simple represenatations. □

2.1. **An infinite-dimensional simple representation.** There are also infinite dimensional simple $\mathfrak{sl}_2(k)$-representations. $E := X_0\partial_1, F := X_1\partial_0, H := X_0\partial_0 - X_1\partial_1$, still act by derivations on $k(X_0, X_1)$, the collection of rational functions

$$k(X_0, X_1) = \{\frac{P}{Q}; P, Q \in k[X_0, X_1], Q \neq 0\}.$$

Let V be the smallest $\mathfrak{sl}_2(k)$-subrepresentation containing $v_0 := \frac{1}{X_0}$. Define for $i \geq 1$ by induction $v_i = F \cdot v_{i-1}$. Then necessarily $v_i \in V$. For example

$$v_1 = X_1\partial_0(\frac{1}{X_0}) = -\frac{X_1}{X_0^2}, v_2 = X_1\partial_0(\frac{-X_1}{X_0^2}) = 2\frac{X_1^2}{X_0^3},$$

and by induction

$$v_i = (-1)^i i! \frac{X_1^i}{X_0^{i+1}}.$$

Then

$$H \cdot v_i = (X_0\partial_0 - X_1\partial_1)\left((-1)^i i! \frac{X_1^i}{X_0^{i+1}}\right) = (-(i+1) - i)\left((-1)^i i! \frac{X_1^i}{X_0^{i+1}}\right) = -(2i+1)v_i$$

so all v_i are H-eigenvectors with eigenvalue $-(2i+1)$, and so they are all linearly independent. Next $E \cdot v_0 = 0$ and for $i \geq 1$

$$E \cdot v_i = X_0\partial_1\left((-1)^i i! \frac{X_1^i}{X_0^{i+1}}\right) = i\left((-1)^i i! \frac{X_1^{i-1}}{X_0^i}\right) = -i^2 v_{i-1}.$$

Finally

$$\begin{aligned}\Delta \cdot v_i &= E\cdot F\cdot v_i + F\cdot E\cdot v_i + \frac{1}{2}H\cdot H\cdot v_i \\ &= E\cdot v_{i+1} - i^2 F\cdot v_{i-1} - \frac{2i+1}{2}H\cdot v_i \\ &= -(i+1)^2 v_i - i^2 v_i + \frac{(2i+1)^2}{2} v_i \\ &= -\frac{1}{2}v_i,\end{aligned}$$

shows that Δ acts on V by multiplication by $-\frac{1}{2}$.

So the space U with basis $\{v_i; i=0,1,2,\ldots\}$ is indeed an $\mathfrak{sl}_2(k)$-representation, and the smallest sub-representation of $k(X_0,X_1)$ containing $\frac{1}{X_0}$.

Lemma 2.3. *$V=\frac{1}{X_0}k[\frac{X_1}{X_0}]$ is a simple infinite dimensional $\mathfrak{sl}_2(k)$-representation. All H-eigenspaces are one-dimensional; the H-eigenspace with the highest eigenvalue -1 is the only eigenspace of E (with 0 eigenvalue).*

Proof. The proof is the same as in the finite case. Let $U \subset V$ be a non-zero subrepresentation. So there is a non-zero $u \in U$. By definition u is a linear combination of finitely many of the basis vectors, say $u=\sum_{i=0}^m c_i v_i$, with $c_m \neq 0$. Now $E^m(u) \in U$ is of the form cv_0 for some non-zero $c \in k$. Hence $v_0 \in U$ and therefore all $u_i \in U$ and so $U=V$. □

Remark. In a dual fashion it can be shown that $V' := \frac{1}{X_1}k[\frac{X_0}{X_1}]$ is a simple infinite dimensional $\mathfrak{sl}_2(k)$-module whose H-eigenspace with *lowest* eigenvalue $+1$ is the only eigenspace of F (with 0 eigenvalue). V is an example of a simple *highest weight* representation, and V' is an example of a simple *lowest weight* representation.

Put $x=\frac{X_1}{X_0}$ and consider the subalgebra $R=k[x]$ of $k(X_0,X_1)$. Our Lie algebra $\mathfrak{sl}_2(k)$ still acts on the polynomial ring $k[x]$ by derivations, explicitly:

$$E=\frac{d}{dx}, F=-x^2\frac{d}{dx}, H=-2x\frac{d}{dx}, \Delta=0.$$

In fact, it is the smallest $\mathfrak{sl}_2(k)$-subrepresentation of $k(X_0,X_1)$ containing $x=\frac{X_1}{X_0}$, with basis

$$\{x^i; i=0,1,2,\ldots\}.$$

But it is not simple, since it contains the trivial sub-representation of constant functions $k\cdot 1$. There is no complementary subrepresentation U such that $R=k\cdot 1\oplus U$, since by a similar method as in the proof above, any non-zero $\mathfrak{sl}_2(k)$-submodule U contains 1 (which you can see by repeatedly using the operator $E=\frac{d}{dx}$). So in the infinite case it is no longer true that every subrepresentation has a complement.

Let us try to exponentiate the $\mathfrak{sl}_2(k)$-action to obtain an $\mathrm{SL}_2(k)$-action. For E there is no problem.

$$\begin{aligned}\exp(tE)(f) &:= 1(f) + t\frac{d}{dx}(f) + \frac{t^2}{2}\frac{d^2}{dx^2}(f) + \ldots + \frac{t^i}{i!}\frac{d^i}{dx^i}(f) + \ldots \\ &= f(x) + tf'(x) + \frac{t^2}{2}f''(x) + \ldots + \frac{t^i}{i!}f^{(i)}(x)) + \ldots \\ &= f(x+t)\end{aligned}$$

by Taylor's formula; so $\exp(t(E)(f(X)) = f(X+t)$. We take $k = \mathbb{C}$, then for any exponent $r \geq 1$

$$\begin{aligned}\exp(tH)(x^r) &= (1 + (-2rt) + \frac{(-2rt)^2}{2} + \ldots + \frac{(-2rt)^i}{i!} + \ldots)x^r \\ &= (\exp(-2t)x)^r\end{aligned}$$

or $\exp(tH)(f(x)) = f(\exp(-2t)x)$ makes sense for polynomials $f(x) \in \mathbb{C}[x]$. By induction $F^i(x) = (-1)^i i! x^{i+1}$ and

$$\exp(tF)(x) = \sum_{i=0}^{\infty} \frac{(tF)^i(x)}{i!} = \sum_{i=0}^{\infty}(-t)^i x^{i+1} = x\sum_{i=0}^{\infty}(-tx)^i = \frac{x}{tx+1}$$

which clearly is not a polynomial. So we did not succeed in exponentiating the action. But being so close we should not give up, but enlarge our ring. One possibility is the following.

Let $\mathcal{O}$ be the ring of analytic functions $f(z)$ on the upper half plane $\{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}$, then for every real number $t \in \mathbb{R}$ the functions $f(z+t)$, $f(\exp(-2t)z)$ and $f(\frac{z}{tz+1})$ are also analytic on the upper half plane, i.e., remain in $\mathcal{O}$ (since $t \in \mathbb{R}$). Or in that case we do get an $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{O}$ by algebra automorphisms

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) := f(\frac{az+b}{cz+d}),$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $f(z) \in \mathcal{O}$. By differentiating this action we get our $\mathfrak{sl}_2(\mathbb{R})$-action back again, but extended to $\mathcal{O}$:

$$E = \frac{d}{dz}, F = -z^2\frac{d}{dz}, H = -2z\frac{d}{dz}.$$

This action is famous in number theory.

2.2. **Extension of complete reductivity to semi-simple Lie algebras.** Working with $\mathfrak{sl}_2(k)$ certainly has the advantage of being explicit. But most of the results proved above remain true in some form for semi-simple Lie algebras. The most important result being the complete reducibility of any representation.

Theorem 2.3 (Weyl)**.** *Let $\mathfrak{g}$ be a semi-simple Lie algebra. If $U \subset V$ is a sub-representation, then there is a sub-representation $U' \subset V$ such that $V = U \oplus U'$.*

Proof. It suffices to prove the special case where U has codimension one, as in the $\mathfrak{sl}_2(k)$-case. Since $\mathfrak{g}$ is semi-simple it acts trivially on all one-dimensional representations, in particular it acts trivially of V/U. We will prove this special case of Weyl's theorem by induction on the dimension of V. If $\dim V = 1$ then $U = 0$ and we can take $U' = V$.

Suppose U is not a simple $\mathfrak{g}$-representation. Then there is a proper sub-representation $W \subset U$. And $U/W \subset V/W$ is a sub-representation of codimension 1, so by induction there is a $v_0 \in V$, not in U, such that $k\overline{v_0} = (W + kv_0)/W$ is a complementary sub-representation, or $V/W = U/W \oplus (W + kv_0)/W$ is a direct sum of representations. In particular $(W + kv_0)$ is itself a representation with sub-representation W of codimension one. We can use induction again and conclude there is $v \in W + kv_0$ such that kv is a complementary one dimensional representation or

$W + kv_0 = W \oplus kv$. Now v is not in U, since otherwise $W + kv_0$ would be part of U but this is a contradiction with the choice of v_0. So we can take kv as the complementary module: $V = U \oplus kv$.

We can therefore suppose that U is simple. Suppose first that U is the trivial one-dimensional representation. Also V/U is the trivial representation. For any $v \in V$ and $X \in \mathfrak{g}$ we have $X \cdot v \in U$, and so for any $Y \in \mathfrak{g}$: $Y \cdot X \cdot v = 0$ and so $[Y, X] \cdot v = 0$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ it follows that $\mathfrak{g}$ acts trivially on the whole of V. Now let $v \in V$ be any element not in U, then kv is also a representation and $U \oplus kv = V$ and we proved this special case.

Now suppose that U is simple but not trivial. If $\rho : \mathfrak{g} \to \mathrm{End}_k(V)$ is the homomorphism associated to V. We can replace $\mathfrak{g}$ by $\rho(\mathfrak{g})$, since the kernel of ρ acts trivially anyway. Also $\rho(\mathfrak{g})$ is semi-simple too (the kernel of ρ, like any ideal, is the direct sum of some of the simple ideals in the direct sum of $\mathfrak{g}$ into simple ideals).

So we can suppose that $\mathfrak{g}$ acts faithfully on V. By Corollary 1.5 the form $B_V(\cdot,\cdot)$ is non-degenerate. Let $X_1, \ldots, X_m$ be a basis of $\mathfrak{g}$ with dual basis $Y_1, \ldots, Y_m$ with respect to $B_V(\cdot,\cdot)$. We get a Casimir operator Δ acting on V by $\Delta \cdot v = \sum_i X_i \cdot Y_i \cdot v$.

We prove that it commutes with the $\mathfrak{g}$-action. Let $X \in \mathfrak{g}$. There are $a_{ij}, b_{ij} \in k$ such that

$$[X, X_i] = \sum_j a_{ij} X_j; \ [X, Y_i] = \sum_j b_{ij} Y_j,$$

so

$$a_{ij} = B_V([X, X_i], Y_j) = -B_V(X_i, [X, Y_j) = -b_{ji}.$$

We check

$$\begin{aligned}
(X \cdot \Delta - \Delta \cdot X) \cdot v &= \sum_i (X \cdot X_i \cdot Y_i - X_i \cdot Y_i \cdot X) \cdot v \\
&= \sum_i ([X, X_i] \cdot Y_i - X_i \cdot [Y_i, X]) \cdot v \\
&= \sum_{i,j} (a_{ij} X_j \cdot Y_i + b_{ij} X_i \cdot Y_i) \cdot v \\
&= \sum_{i,j} (a_{ij} X_j \cdot Y_i + b_{ji} X_j \cdot Y_i) \cdot v \\
&= 0
\end{aligned}$$

So the map

$$\Delta : V \to V : v \mapsto \Delta \cdot v$$

is a homomorphism of representations. Since $\mathfrak{g}$ acts trivially on V/U, also Δ acts trivially on V/U, i.e. $\Delta \cdot V \subset U$. Its trace is

$$\mathrm{tr}(\Delta) = \sum_i \mathrm{tr}_V(\rho(X_i)\rho(Y_i)) = \sum_i B_V(X_i, Y_i) = \dim V.$$

Since $B_V = B_U + B_{V/U}$ and $\mathfrak{g}$ acts trivially on V/U we get $B_V = B_U$. So Δ does not act trivially on U (since its trace is $\dim \mathfrak{g}$) so $\Delta \cdot U$ is a non-zero sub-representation of the simple representation U, so the image of Δ is U. It follows that the kernel of Δ, say U', is a one-dimensional sub-represntation with zero-intersection with U. We get the complement we were looking for: $V = U \oplus U'$. This finishes the proof of Weyl's theorem. □

3. Complexification of simple Lie algebras

For any Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semi-simple, i.e., isomorphic to a direct sum of ideals, each of which is simple. We would like to have more information on simple Lie algebras. In fact, there is a lot of information available. And there is a classification : we know them all.

3.1. **Complexification.** Let us recall the process of complexification of a vector space. Let V be a real vector space. We define a new real vector space $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ as the set of formal expressions

$$V^{\mathbb{C}} := \{v \otimes 1 + w \otimes i;\ v, w \in V\}$$

(where $i \in \mathbb{C}$ is as usual a square root of -1). In particular, $v_1 \otimes 1 + 1 \otimes i = v_2 \otimes 1 + w_2 \otimes i$ if and only if $v_1 = v_2$ and $w_1 = w_2$. We define the addition

$$(v_1 \otimes 1 + w_1 \otimes i) + (v_2 \otimes 1 + w_2 \otimes i) := (v_1 + v_2) \otimes 1 + (w_1 + w_2) \otimes i$$

and multiplication by a real number $\lambda \in \mathbb{R}$:

$$\lambda(v \otimes 1 + w \otimes i) := \lambda v \otimes 1 + \lambda w \otimes i.$$

With this structure $V^{\mathbb{C}}$ becomes a real vector space isomorphic to $V \oplus V$. Now we *define* for a complex number $z = a + bi$ $(a, b \in \mathbb{R})$ and $v \in V$:

$$v \otimes z := av \otimes 1 + bv \otimes i \in V^{\mathbb{C}}$$

Then

$$v \otimes \lambda z = \lambda v \otimes z, (v_1 + v_2) \otimes z = (v_1 \otimes z) + (v_2 \otimes z), v \otimes (z_1 + z_2) = (v \otimes z_1) + (v \otimes z_2),$$

for $z, z_1, z_2 \in \mathbb{C}$, $\lambda \in \mathbb{R}$, $v, v_1, v_2 \in V$.

Finally we *define* multiplication by a complex number as

$$\begin{aligned} z \cdot (v \otimes 1 + w \otimes i) := v \otimes z + w \otimes zi \\ &= (av \otimes 1 + bv \otimes i) + (aw \otimes i - bw \otimes 1) \\ &= (av - bw) \otimes 1 + (bv + aw) \otimes i. \end{aligned}$$

So in particular, if $z_1, z_2 \in \mathbb{C}$ and $v \in V$ then

$$z_1 \cdot (v \otimes z_2) = v \otimes z_1 z_2.$$

With this addition and complex multiplication $V^{\mathbb{C}}$ becomes a *complex* vector space. It comes equipped with a map induced by the complex conjugation map

$$\theta : V^{\mathbb{C}} \to V^{\mathbb{C}}; v \otimes 1 + w \otimes i \mapsto v \otimes 1 - w \otimes i,$$

$v, w \in V$. So with the usual complex conjugation $\theta(v \otimes z) = v \otimes \overline{z}$, with $v \in V, z \in \mathbb{C}$. So θ is an involution ($\theta^2 = \mathbf{1}$), an isomorphism as real vector space, but $\theta(zu) = \overline{z}\theta(u)$, for $u \in V^{\mathbb{C}}, z \in \mathbb{C}$. The fixed points for θ is the "real part" i.e. $V \otimes 1$.

We remark that an arbitrary complex vector space does not come equipped automatically with such a map θ !

But if a complex vector space comes equipped with such a θ, then it comes from complexification of a specific real vector space. Suppose W is a (finite dimensional) complex vector space together with a map $\theta : W \to W$, which is an involution, real linear and $\theta(zw) = \overline{z}\theta(w)$, for $z \in \mathbb{C}$ and

$w \in W$. Take $V = \{w \in W; \theta(w) = w\}$. Then V is a real sub vector-space of W, iV is also a real sub vector space and $V \cap iV = 0$ (since if $v = iw$, for $v, w \in W$, then $v = \theta v = \theta iw = -iw = -v$ and so $v = 0$). Let $w \in W$, then $w + \theta(w)$ and $-iw + i\theta(w)$ are in V, since

$$\theta(w + \theta(w)) = \theta(w) + \theta(\theta(w)) = w + \theta(w); \theta(-iw + i\theta(w)) = i\theta(w) - iw.$$

So

$$w = \frac{1}{2}(w + \theta(w)) + i\frac{1}{2}(-iw + i\theta(w)) \in V + iV$$

and so $W = V \oplus iV$. Then

$$V^{\mathbb{C}} \simeq W : (v_1 \otimes 1 + v_2 \otimes i) \mapsto v_1 + iv_2$$

gives an isomorphism as complex vector spaces.

If $v_1, \ldots, v_n$ is a real basis of V, then $v_1 \otimes 1, v_2 \otimes 1, \ldots, v_n \otimes 1$ is a basis for $V^{\mathbb{C}}$ as complex vector space.

Sometimes we identify V with $V \otimes 1$, then $iV = V \otimes i$ and $V^{\mathbb{C}} = V \oplus iV$. And then a real basis for V is at the same time a complex basis for $V^{\mathbb{C}}$.

We will avoid this identification when V is a complex vector space seen as a real vector space. In that case we have to be careful to distinguish between the old complex multiplication and the new complex multiplication, say in that case $z_1 v \otimes z_2$ (using the old complex multiplication) is not the same as $z_1 \cdot (v \otimes z_2) = v \otimes z_1 z_2$ (the new complex multiplication), where $z_1, z_2 \in \mathbb{C}, v \in V$.

If $\mathfrak{g}$ is a real Lie algebra, then $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra, with Lie bracket

$$[X_1 \otimes z_1, X_2 \otimes z_2] := [X_1, X_2] \otimes z_1 z_2,$$

where $X_1, X_2 \in \mathfrak{g}$, $z_1, z_2 \in \mathbb{C}$. And so

$$\theta([Z_1, Z_2]) = [\theta(Z_1), \theta(Z_2)],$$

for $Z_1, Z_2 \in \mathfrak{g}^{\mathbb{C}}$.

On the other hand, let $\mathfrak{k}$ be a complex Lie algebra together with an involution $\theta : \mathfrak{k} \to \mathfrak{k}$ which is an automorphism of order two of real Lie algebras, such that $\theta(uZ) = \overline{u}\theta(Z)$ for $u \in \mathbb{C}$ and $Z \in \mathfrak{k}$. Then $\mathfrak{g} = \{Z \in \mathfrak{k}; \theta(Z) = Z\}$ is a real sub-algebra and

$$\mathfrak{g}^{\mathbb{C}} \to \mathfrak{k} : X \otimes 1 + Y \otimes i \mapsto X + iY$$

with $X, Y \in \mathfrak{g}$ is an isomorphism of complex Lie algebras.

Similarly, if V is a real representation of the real Lie algebra $\mathfrak{g}$, then $V^{\mathbb{C}}$ is a complex representation of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, with multiplication

$$(X \otimes z_1) \cdot (v \otimes z_2) := (X \cdot v) \otimes z_1 z_2.$$

The two natural complex conjugation maps θ are related by

$$\theta(Xu) = \theta(X)\theta(u),$$

where $X \in \mathfrak{g}^{\mathbb{C}}$ and $u \in V^{\mathbb{C}}$.

For the associated bilinear forms we get

$$B_{V^{\mathbb{C}}}(X_1 \otimes z_1, X_2 \otimes z_2) = z_1 z_2 B_V(X_1, X_2) \in \mathbb{C}$$

for $X_1, X_2 \in \mathfrak{g}$, $z_1, z_2 \in V$. A real basis of $\mathfrak{g}$ can be identified with a complex basis of $\mathfrak{g}^{\mathbb{C}}$, so the associated Gramm-matrices are the same. Therefore B_V is non-degenerate on $\mathfrak{g}$ if and only if $B_{V^{\mathbb{C}}}$ is non-degenerate on $\mathfrak{g}^C$. In particular, $\mathfrak{g}$ is a real semisimple Lie algebra if and only if $\mathfrak{g}^{\mathbb{C}}$ is a complex semisimple Lie algebra, by Cartan's theorem.

3.2. **Complexification of real simple Lie algebras.** If $\mathfrak{g}$ is a simple real Lie algebra, then its complexification $\mathfrak{g}^{\mathbb{C}}$ is not necessarily a simple complex Lie algebra, but at least it is semi-simple, hence a direct sum of simple ideals.

Example 3.1. $\mathfrak{sl}_2(\mathbb{R})^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{su}_2^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$ but $\mathfrak{sl}_2(\mathbb{C})^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

Proposition 3.1. *Suppose that $\mathfrak{g}$ is a simple complex Lie algebra, but we consider it as a real Lie algebra.*

(i) $\mathfrak{g}$ is also simple considered as a real Lie algebra.

(ii) There is an isomorphism of real Lie algebras

$$\mathfrak{g}^{\mathbb{C}} \to \mathfrak{g} \oplus \mathfrak{g} : (A \otimes 1 + B \otimes i) \mapsto (A + iB, A - iB).$$

Its inverse is $(X, Y) \mapsto \left(\frac{1}{2}(X+Y) \otimes 1\right) + \left(\frac{1}{2i}(X-Y) \otimes i\right)$.

Complex conjugation θ for $\mathfrak{g}^{\mathbb{C}}$ corresponds to the map

$$\mathfrak{g} \oplus \mathfrak{g} : (X, Y) \mapsto (Y, X).$$

Multiplication by the new i on $\mathfrak{g}^{\mathbb{C}}$ (on the rightsand side of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$) corresponds to the map

$$\mathfrak{g} \oplus \mathfrak{g} : (X, Y) \mapsto (iX, -iY).$$

Proof. (i) Suppose $\mathfrak{a}$ is a minimal non-zero real ideal of $\mathfrak{g}$. So it is a real sub vector-space such that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{g}$. Now $\mathfrak{a} + i\mathfrak{a}$ is a complex ideal of $\mathfrak{g}$, hence $\mathfrak{a} + i\mathfrak{a} = \mathfrak{g}$. Also $[\mathfrak{g}, i\mathfrak{a}] = i[\mathfrak{g}, \mathfrak{a}] \subset i\mathfrak{a}$ shows that $i\mathfrak{a}$ is also a real ideal. If $i\mathfrak{a} = \mathfrak{a}$ then $\mathfrak{a}$ is even a complex ideal, hence $\mathfrak{a} = \mathfrak{g}$. If not, by minimality of $\mathfrak{a}$ we get $\mathfrak{a} \cap i\mathfrak{a} = 0$ and $\mathfrak{g} = \mathfrak{a} \oplus i\mathfrak{a}$, a direct sum of real ideals. Let $A, B \in \mathfrak{a}$ then $[iA, B] = i[A, B] \in \mathfrak{a} \cap i\mathfrak{a} = 0$ so $[A, B] = 0$ and hence also $\mathfrak{g}$ is abelian. But $\mathfrak{g}$ is simple. Contradiction.

(ii) It is certainly real linear. Since

$$[A \otimes 1 + B \otimes i, A' \otimes 1 + B' \otimes i] = ([A, A'] - [B, B']) \otimes 1 + ([A, B'] + [B, A']) \otimes i$$

is mapped to

$$(([A, A'] - [B, B']) - i([A, B'] + [B, A']), ([A, A'] - [B, B']) + i([A, B'] + [B, A']) =$$

$$= ([A - iB, A' - iB'], [A + iB, A' + iB']),$$

it follows that the bracket is preserved. That the two maps are inverses to each other is checked directly. □

We have a converse too.

Proposition 3.2. *Suppose $\mathfrak{g}$ is a simple real Lie algebra such that $\mathfrak{g}^{\mathbb{C}}$ is not simple. Then $\mathfrak{g}$ is a simple complex Lie algebra, considered as a real Lie algebra.*

Proof. Let $\mathfrak{a}$ be a simple complex ideal of $\mathfrak{g}^{\mathbb{C}}$, then $\theta(\mathfrak{a})$ is also a simple complex ideal of $\mathfrak{g}^{\mathbb{C}}$. Put $\mathfrak{a}_0 = \mathfrak{a} \cap (\mathfrak{g} \otimes 1)$. Then $\mathfrak{a}_0$ is an ideal of the simple real Lie algebra $\mathfrak{g} \otimes 1 \simeq \mathfrak{g}$. So $\mathfrak{a}_0 = 0$ or $\mathfrak{a}_0 = \mathfrak{g} \otimes 1$.

In case $\mathfrak{a}_0 = \mathfrak{g} \otimes 1$, then also $\mathfrak{g} \otimes i \subset \mathfrak{a}$ and so $\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}$. Which is a contradiction, since $\mathfrak{g}^{\mathbb{C}}$ is not simple. So $\mathfrak{a}_0 = 0$.

Let $X \otimes 1 + Y \otimes i \in \mathfrak{a} \cap \theta\mathfrak{a}$ then

$$\frac{1}{2}\left(X \otimes 1 + Y \otimes i + X \otimes 1 - Y \otimes i\right) = X \otimes 1 \in \mathfrak{a}_0$$

and so $X = 0$ and $-i(Y \otimes i) = Y \otimes 1 \in \mathfrak{a}_0$ and so $Y = 0$. We conclude that $\mathfrak{a} \cap \theta\mathfrak{a} = 0$.

Let $X \otimes 1 + Y \otimes i \in \mathfrak{a}$ be non-zero, then also $X \otimes i - Y \otimes 1 \in \mathfrak{a}$ and both $X \otimes 1 - Y \otimes i$ and $-X \otimes i - Y \otimes 1 \in \theta\mathfrak{a}$ and so $2X \otimes 1$ and $2Y \otimes 1 \in (\mathfrak{a} \oplus \theta\mathfrak{a}) \cap \mathfrak{g} \otimes 1$, and so $(\mathfrak{a} \oplus \theta\mathfrak{a}) \cap (\mathfrak{g} \otimes 1) \neq 0$. From the simplicity of $\mathfrak{g}$ it follows that $(\mathfrak{g} \otimes 1) \subset \mathfrak{a} \oplus \theta\mathfrak{a}$. And so we get a direct sum of complex ideals

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{a} \oplus \theta\mathfrak{a}.$$

The complex linear map

$$\mathfrak{a}^{\mathbb{C}} \to \mathfrak{a} \oplus \theta\mathfrak{a} : (A \otimes 1 + B \otimes i) \mapsto (A + iB, \theta(A - iB))$$

has inverse linear map $(X, Y) \mapsto (\frac{1}{2}(X + \theta(Y)) \otimes 1) + (\frac{1}{2i}(X - \theta(Y)) \otimes i)$. It is an isomorphism of complex Lie algebras.

We also have a homomorphism of real Lie algebras

$$\alpha : \mathfrak{a} \to \mathfrak{g} \otimes 1 : Z \mapsto (Z + \theta Z)$$

since

$$\alpha([Z, W]) = ([Z, W] + \theta[Z, W]) = [Z + \theta Z, W + \theta W] - [Z, \theta W] - [\theta Z, W] = [\alpha(Z), \alpha(W)].$$

Let $X \otimes 1 \in \mathfrak{g} \otimes 1$. There are unique $X_1 \otimes 1 + Y_1 \otimes i$ and $X_2 \otimes 1 + Y_2 \otimes i \in \mathfrak{a}$ such that

$$X \otimes 1 = (X_1 \otimes 1 + Y_1 \otimes i) + (X_2 \otimes 1 - Y_2 \otimes i) = (X_1 + X_2) \otimes 1 + (Y_1 - Y_2) \otimes i.$$

So $Y_2 = Y_1$. But then

$$(X_1 \otimes 1 + Y_1 \otimes i) - (X_2 \otimes 1 + Y_2 \otimes i) = (X_1 - X_2) \otimes 1 \in \mathfrak{a}_0$$

hence $X_1 = X_2 = \frac{1}{2}X$. And for any $X \in \mathfrak{g}$ there is a unique $Z \in \mathfrak{a}$ such that $X = Z + \theta Z = \alpha(Z)$.

Or in other words: $\alpha : \mathfrak{a} \to \mathfrak{g}$ is an isomorphism of real Lie algebras, and the complexifications of $\mathfrak{a}$ and $\mathfrak{g}$ are isomorphic. □

Suppose we know the classification of complex simple Lie algebras. They are also real simple Lie algebras. If we also know all the possible θ's on the complex simple Lie algebras, we can deduce the simple real Lie algebras.

Up to only five exceptions any complex simple Lie algebra is a member of one of three families

$$\mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C}).$$

The last of the three families is associated to non-degenerate anti-symmetric complex bilinear forms, in the same way as the second family is associated to non-degenerate symmetric complex bilinear forms.

3.3. **Bilinear forms on a complex simple Lie algebra.** Being a simple complex Lie algebra $\mathfrak{g}$ implies that there is up to a constant only one invariant bilinear form, the Killing form. A $\mathbb{C}$-bilinear form C on $\mathfrak{g}$ is called $\mathfrak{g}$-invariant if for all $X, Y, Z \in \mathfrak{g}$:

$$C(\mathrm{ad}(X)(Y), Z) + C(Y, \mathrm{ad}(X)(Z)) = 0$$

or $C(Y, [X, Z]) = C([Y, X], Z)$. If the form comes from a complex representation, it has this invariancy.

Proposition 3.3. *Let $\mathfrak{g}$ be a simple complex Lie algebra. For any $\mathfrak{g}$-invariant bilinear form C on $\mathfrak{g}$ there exists a $c \in \mathbb{C}$ such that for all $X, Y \in \mathfrak{g}$ we have $C(X, Y) = cB(X, Y)$, where B is the Killing form.*

Proof. For any $X \in \mathfrak{g}$ we get a linear form $C(X, \cdot) \in \mathfrak{g}^*$. Since the Killing form is non-degenerate there is a unique $\alpha(X) \in \mathfrak{g}$ such that $C(X, \cdot) = B(\alpha(X), \cdot)$. The map $\alpha : \mathfrak{g} \to \mathfrak{g}$ is clearly linear, but we have more. Since both forms are $\mathfrak{g}$-invariant, we have for all $Y, Z \in \mathfrak{g}$

$$\begin{aligned} B(\alpha(\mathrm{ad}(Y)(X)), Z) &= C(\mathrm{ad}(Y)(X), Z) = -C(X, \mathrm{ad}(Y)(Z)) = -B(\alpha(X), \mathrm{ad}(Y)(Z)) \\ &= B(\mathrm{ad}(Y)(\alpha(X)), Z), \end{aligned}$$

so $B(\alpha(\mathrm{ad}(Y)(X)), \cdot) = B(\mathrm{ad}(Y)(\alpha(X)), \cdot)$, and by unicity

$$\alpha(\mathrm{ad}(Y)(X)) = \mathrm{ad}(Y)(\alpha(X)).$$

In other terms the map $\alpha : \mathfrak{g} \to \mathfrak{g}$ is not quite a homomorphism of Lie algebras, but it is a morphsim of $\mathfrak{g}$-representations. Its kernel is a sub representation of $\mathfrak{g}$, hence an ideal. If the kernel is $\mathfrak{g}$ then α is the zero-map and the form C is zero. As constant we can take $c = 0$. If not, then α is injective, hence α is an automorphism of $\mathfrak{g}$-representations. There is an eigenvalue $c \in \mathbb{C}$ for α acting on $\mathfrak{g}$. Then $(\alpha - c\mathbf{1})$ is also an endomorphism of $\mathfrak{g}$-representations, so its kernel is a sub-representation hence either 0 or $\mathfrak{g}$, by simplicity of the Lie algebra. This kernel is just the eigenspace of eigenvalue c, which is non-zero, hence the whole space. This means that $\alpha(X) = cX$ for all $X \in \mathfrak{g}$. Or $C(X, Y) = B(cX, Y) = cB(X, Y)$. □

Remark. It also follows there is up to scalar only one Casimir operator associated to a simple complex Lie algebra (for some more information, see proof of Weyl's theorem).

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