Homework 5
Math 16210-Section 51

Due: Tuesday March 10th

Exercise 1. **1.** Prove that if $f : [a, b] \to \mathbb{R}$ is a continuous function such that $\int_a^b |f| = 0$ then $f = 0$.

2. Find all continuous functions $f : [0, 1] \to [0, 1]$ such that $\int_0^1 f(t)\mathrm{d}t = \int_0^1 f(t)^2 \mathrm{d}t$.

Exercise 2. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$. Suppose that there exists $x \in [a, b]$ such that $f(x) \neq 0$ and that there exists $n \in \mathbb{N}$ such that for all $k \leqslant n$, $\int_a^b t^k f(t)\mathrm{d}t = 0$. We want to prove that there exists $n + 1$ distinct points in $[a, b]$ where f vanishes and changes sign.

1. Study the case $n = 0$ and $n = 1$.

2. Prove the statement for all n.

Exercise 3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. For all $x \in \mathbb{R}$ we define $g(x) = \int_0^1 f(t)\mathrm{e}^{tx}\mathrm{d}t$. Prove that g is continuous on $\mathbb{R}$.

Exercise 4. Let $f : [0, 1] \to \mathbb{R}$ be a strictly increasing function such that $f(0) = 0$ and $f(1) = 1$. Prove that $\lim_{n\to+\infty} \int_0^1 (f(t))^n \mathrm{d}t = 0$

Exercise 5. Find the limits of the following sequences

1. $u_n = n\left(\frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+n)^2}\right)$.

2. $u_n = \sqrt[n]{\left(1 + \left(\frac{1}{n}\right)^2\right)\left(1 + \left(\frac{2}{n}\right)^2\right)\cdots\left(1 + \left(\frac{n}{n}\right)^2\right)}$.

3. $u_n = \frac{1}{n}\prod_{k=1}^{n}(k+n)^{1/n}$.

4. $u_n = \sum_{p=n}^{2n}\frac{1}{p}$.

Exercise 6. Let $f : [a, b] \to \mathbb{R}$ be continuous and $g : \mathbb{R} \to \mathbb{R}$ be continuous and convex. Show that

$$g\left(\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right) \leqslant \frac{1}{b-a}\int_a^b g(f(t))\mathrm{d}t.$$

Exercise 7. Let $f : [a, b] \to \mathbb{R}$ be $C^1([a, b])$. We define

$$R_n(f) := \frac{b-a}{n}\sum_{k=0}^{n-1} f\left(a + \frac{k(b-a)}{n}\right) \quad \text{and} \quad M_1 := \sup\{|f'(x)| \mid x \in [a, b]\}.$$

1. Prove that

$$\int_a^b |f(t) - f(\alpha)|\mathrm{d}t \leqslant M_1\frac{(\beta-\alpha)^2}{2}.$$

2. For all $n \geqslant 1$, show that

$$\left|\int_a^b f(t)\mathrm{d}t - R_n(f)\right| \leqslant \frac{M_1(b-a)^2}{2n}.$$

3. Find an approximate value of $\int_0^1 \mathrm{e}^{-x^2}\mathrm{d}x$ with an error less than 10^{-1}.

Georg Friedrich Bernhard Riemann
(1826–1866)

Johan Ludvig William Valdemar Jensen
(1859–1925)