Michel Rolle
(1652–1719)

Homework 4
Math 16210-Section 50

Due: Tuesday March 3rd

Jean Gaston Darboux
(1842–1917)

Exercise 1. Let f be defined on a interval containing 0 which is continuous on I. We suppose also that

$$\lim_{x\to 0}\frac{f(2x)-f(x)}{x}=0.$$

Prove that f is differentiable at 0.

Exercise 2. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable such that $f(0)=0$. Show that $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right)$ exists and compute it.

Exercise 3. **1.** Show that for all $x>0$ we have $\frac{1}{x+1}<\ln(1+x)-\ln(x)<\frac{1}{x}$.

2. Compute for all $k\in\mathbb{N}\setminus\{1\}$, $\lim_{n\to\infty}\sum_{p=n+1}^{kn}\frac{1}{p}$.

Exercise 4. Let $n\in\mathbb{N}$ and $f:I\to\mathbb{R}$ be a function differentiable n times and such that its n-th derivative $f^{(n)}$ is continuous. We suppose that f vanishes at $n+1$ distinct points in I: there exist $x_0,\dots,x_n$ distinct points in I such that $f(x_i)=0$.

1. Show that the n-th derivative of f vanishes at, at least, one point in I.

2. Let $\alpha\in\mathbb{R}$. Show that the $(n-1)$-th derivative of $f'+\alpha f$ vanishes at, at least, one point in I.
Hint: One can introduce an auxiliary function

Exercise 5. Let $n\in\mathbb{N}$ and let $f:[a,b]\to\mathbb{R}$ be such that $f,f',f'',\dots,f^{(n)}$ are continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b). Prove that if $x_0\in[a,b]$, then for any $x\in[a,b]$ there exists a c between x_0 and x such that

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2}f''(x_0)+\cdots+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(c).$$

Exercise 6. Prove that we have for all $x\in\mathbb{R}$,

$$\arctan(\mathrm{e}^x)=\arctan\left(\tanh\left(\frac{x}{2}\right)\right)+\frac{\pi}{4},\quad \text{we recall } \tanh(x)=\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}.$$

Exercise 7. Let I be an open interval of $\mathbb{R}$ and f be differentiable on I. We want to prove that the intermediate value theorem applies to f' (even if f' is not continuous!).

1. Let $(a,b)\in I^2$ such that $f'(a)<f'(b)$ and $z\in(f'(a),f'(b))$. Show that there exists $\alpha>0$ such that for all $h\in(0,\alpha)$ we have

$$\frac{1}{h}(f(a+h)-f(a))<z<\frac{1}{h}(f(b+h)-f(b)).$$

2. Show that there exist $h>0$ and $y\in I$ such that

$$y+h\in I\quad\text{and}\quad \frac{1}{h}(f(y+h)-f(y))=z$$

3. Prove that there exists $x\in I$ such that $z=f'(x)$ and that $f'(I)$ is an interval.

4. Let $f(x)=x^2\sin\left(\frac{1}{x}\right)$ on $(0,1]$ and $f(0)=0$. Show that f is differentiable on $[0,1]$. Is f' continuous on $[0,1]$? Find $f'([0,1])$, what can you conclude?

Exercise 8. Let $f:\mathbb{R}\to\mathbb{R}$ be infinitely many times differentiable and bounded.

1. Show that if $f^{(k)}$ admits finitely many zeros then we have $\lim_{x\to\pm\infty}f^{(p)}(x)=0$ for all $p\in\{1,\dots,k-1\}$.

2. Prove that, in this case, for $k\geqslant 2$, $f^{(k)}$ vanishes at, at least, $k-1$ points.

Exercise 9. Let $f:[a,b]\to[a,b]$ be a differentiable function such that there exists $k\in(0,1)$ such that for all $x\in[a,b]$, $|f'(x)|\leqslant k$. Prove that there exists a unique $\gamma\in[a,b]$ such that $f(\gamma)=\gamma$.