Homework 3
Math 16210-Section 50

Due: Tuesday February 18th

Exercise 1. With the (ε, δ) definition of the limit, prove that $\lim_{x \to 1} x^3 = 1$.

Exercise 2. Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ then $|f|$ is continuous at x_0. Is the converse true?

Exercise 3. *On Thomae's function.* Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } x = 0, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p \wedge q = 1 \text{ and } q \geqslant 1. \end{cases}$$

Note that $p \wedge q = 1$ means that p and q are coprime numbers. Prove that f is only continuous on $\mathbb{R} \setminus \mathbb{Q}$ and 0. We should use the fact that both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

Exercise 4. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall (x, y) \in \mathbb{R}^2, \quad f(x + y) = f(x) + f(y).$$

Exercise 5. *On the continuity points of a function.* This exercise is a follow-up on **Exercise 1.** of the previous sheet.

We consider the function $\psi_{\mathbb{Q}} : \mathbb{R} \to \{-1, 1\}$ defined by for $x \in \mathbb{R}$,

$$\psi_{\mathbb{Q}}(x) = 2\mathbb{1}_{\mathbb{Q}}(x) - 1 = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{otherwise.} \end{cases}$$

For a given function $f : \mathbb{R} \to \mathbb{R}$ we denote

$$\Gamma(f) = \{x \in \mathbb{R} \mid f \text{ is continuous at } x\} \quad \text{and} \quad Z(f) = \{x \in \mathbb{R} \mid f(x) = 0\}.$$

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that we have $\Gamma(\psi f) = Z(f)$ (ψf is simply defined by $(\psi f)(x) = \psi(x) f(x)$).

2. The goal of this question it to prove that for every closed set F, there exists a function g such that $\Gamma(g) = F$.

a. We recall the definition of the distance to a set

$$d(x, F) = \inf_{y \in F} |x - y|.$$

Using **Homework 2.**, prove that $x \mapsto d(x, F)$ is continuous. If $x \in F$, give $d(x, F)$.

b. Using **Question 1.** Construct a function $g : \mathbb{R} \to \mathbb{R}$ such that we have $\Gamma(g) = F$.

3. Let $\Omega \subset \mathbb{R}$ be an open set. We define the characteristic function χ_Ω by, for $x \in \mathbb{R}$,

$$\chi_\Omega(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \in \Omega^c. \end{cases}$$

Prove that $\Gamma(\psi \chi_\Omega) = \Omega$.

4. The goal of this question is to prove that if $S \subset \mathbb{R}$ is a G_δ (a countable intersection of open sets) then there exists a function f such that $\Gamma(f) = S$. We suppose that S is a G_δ,

a. Prove the existence of a sequence of open sets $(\Omega_n)_{n\in\mathbb{N}}$ such that

$$\Omega_1 = \mathbb{R}, \quad \forall n \in \mathbb{N}, \Omega_{n+1} \subset \Omega_n, \text{ and } \quad S = \bigcap_{n\in\mathbb{N}} \Omega_n.$$

b. We define $f : \mathbb{R} \to \mathbb{R}$ in the following way: for all $x \in \mathbb{R}$, either $x \in S$ and we set $f(x) = 0$ or there exists an $n \in \mathbb{N}$ such that $x \in \Omega_n \setminus \Omega_{n+1}$ and we set $f(x) = 2^{-n}$. Show that $\Gamma(\psi f) = S$.

5. Finally, the goal of this question is to prove that if $f : \mathbb{R} \to \mathbb{R}$ is any function then $\Gamma(f)$ is a G_δ. Let f be such a function.

a. For any $k \in \mathbb{N}$, we denote

$$\mathcal{O}_k = \{(\alpha, \beta) \subset \mathbb{R} \mid \alpha, \beta \in \mathbb{Q} \text{ and } \beta - \alpha < 2^{-k}\}.$$

Note that here (α, β) denotes the interval, thus $\mathcal{O}_k$ is a set of intervals of $\mathbb{R}$. Prove that for every fixed k, $\mathcal{O}_k$ is countable and that every open set of $\mathbb{R}$ is the union of some elements of $\mathcal{O}_k$. To do so, see that we have, for every open set $U \subset \mathbb{R}$,

$$U = \bigcup_{\substack{V\in\mathcal{O}_k \\ V\subset U}} V.$$

b. Prove that

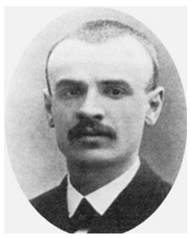
$$\Gamma(f) = \bigcap_{k\in\mathbb{N}} \bigcup_{\Omega\in\mathcal{O}_k} \operatorname{Int}\left(f^{-1}(\Omega)\right)$$

.

c. Using the previous question, prove that $\Gamma(f)$ is a G_δ.

In this exercise you proved the interesting result that a set S is the set of continuity points of a function if, and only if, S is a G_δ. With this theorem, you can try to prove that there does not exist any function f such that $\Gamma(f) = \mathbb{Q}$. If you are brave and want to prove it, you will need *Baire's theorem* (look it up or ask me or don't).

Carl Johannes Thomae
(1840–1921)



René-Louis Baire
(1874–1932)