Homework 2
Math 16210-Section 50

Due: Tuesday February 4th

Exercise 1. *On F_σ and G_δ.* We say that a set is a F_σ if it is the countable union of closed sets and a G_δ if it is the countable intersection of open sets.

1. Show that if $S \subset \mathbb{R}$ is a F_σ then its complement is a G_δ and reciprocally.
2. Give an example of a subset of $\mathbb{R}$ which is a F_σ but which is neither closed nor open.
3. Show that the countable union of F_σ sets is also a F_σ. What analogous property do we have involving G_δ?
4. Let $A \subset \mathbb{R}$ be nonempty. We define for $x \in \mathbb{R}$,

$$\mathrm{d}(x, A) = \inf_{a \in A} |x - a|.$$

   a. Show that we have for every $x, y \in \mathbb{R}$,

$$|\mathrm{d}(x, A) - \mathrm{d}(y, A)| \leqslant |x - y|$$

   b. For $\varepsilon > 0$, we denote

$$A^\varepsilon = \{x \in \mathbb{R} \mid \mathrm{d}(x, A) \leqslant \varepsilon\}.$$

   Give $\bigcap_{\varepsilon > 0} A^\varepsilon$.

5. Let $f : X \to \mathbb{R}$. Show that if f is continuous then for any $U \subset \mathbb{R}$ open, $f^{-1}(U)$ is a F_σ.
6. Show that every closed subset of $\mathbb{R}$ is a G_δ and every open subset is a F_σ.

Exercise 2. Let f, g be two continuous functions from X to $\mathbb{R}$.

1. Show that the set $\{x \in \mathbb{R} \mid f(x) = g(x)\}$ is closed in X.
2. Show that if there exists a set A dense in X such that $f(a) = g(a)$ for all $a \in A$ then $f = g$.

Exercise 3. Let $K \subset \mathbb{R}$ be a compact set and $f : K \to \mathbb{R}$ be a locally bounded function on K: for all $x \in K$, there exists a region R_x with $x \in R_x$ and $M_x > 0$ such that, for all $y \in R_x$, we have $|f(y)| \leqslant M_x$.

1. Show that f is bounded on K.
2. Give an example of $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ such that f is locally bounded on X but not bounded.

Exercise 4. Let $K \subset \mathbb{R}$ be a compact set and $F \subset \mathbb{R}$ be a closed set such that $K \cap F = \emptyset$.

1. Show that there exists an open set U such that $K \subset U$ and $\overline{U} \cap F = \emptyset$.
2. We now suppose that F is compact. Show that there exists an open set V such that $F \subset V$ and $U \cap V = \emptyset$.

Exercise 5. *On the Hausdorff distance.* Let $\mathcal{K}$ be the set of all nonempty compact subsets of $\mathbb{R}$.

1. Let $A \in \mathcal{K}$ and $\alpha, \beta > 0$, show that $A^0 = A$, $A^\alpha = \cap_{r > \alpha} A^r$, and $(A^\alpha)^\beta \subset A^{\alpha + \beta}$ where the notation A^α was introduced in **Exercise 1.**
2. For $A, B \in \mathcal{K}$, we denote $\sigma(A, B) = \inf\{\alpha \mid B \subset A^\alpha\}$. Show that for all $A, B, C \in \mathcal{K}$, we have

$$\sigma(A, B) \leqslant \sigma(A, C) + \sigma(C, B).$$

3. For $A, B \in \mathcal{K}$, we denote $h(A, B) = \max(\sigma(A, B), \sigma(B, A))$. Show that h is a distance on $\mathcal{K}$: for $A, B, C \in \mathcal{K}$ we have

$$h(A, B) = 0 \Rightarrow A = B, \quad h(A, B) = h(B, A), \quad \text{and} \quad h(A, B) \leqslant h(A, C) + h(C, B).$$

4. We denote $\mathcal{H}_s$ to be the set of the nonempty sets of $\mathbb{R}$ which have at most s elements.

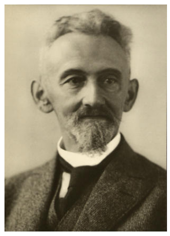
a. Show that we have $H_s \subset \mathcal{K}$.

b. We can define an open set with respect to h in the following sense: $\mathcal{A}$ is open if and only if for very $A \in \mathcal{A}$ there exists $\varepsilon > 0$ such that $B_\varepsilon(A) := \{B \in \mathcal{K} \mid h(A, B) < \varepsilon\} \subset \mathcal{A}$. This is similar to the characterization we gave for open sets last quarter: a set is open if, and only if, for every point in the set there exists a region containing the point included in the set.

Show that $\mathcal{H}_s$ is closed i.e. the complement of an open set.

c. Construct a bijection from $\mathcal{H}_2$ to the half-plane $P = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant y\}$.

Émile Borel
(1871–1956)



Felix Hausdorff
(1868–1942)