Homework 1
Math 16210-Section 50

Due: Tuesday January 21st

Exercise 1. Do the Appendix of Script 7: The Real Numbers are an Ordered Field from Definition 7.30 to Theorem 7.46.

Exercise 2. Let F be a field such that $F \subset \mathbb{Q}$. Show that $F = \mathbb{Q}$.

Exercise 3. Let $\alpha \in \mathbb{N}$ such that $\sqrt{\alpha} \notin \mathbb{Q}$. We denote

$$\mathbb{Q}[\sqrt{\alpha}] = \{a + b\sqrt{\alpha} \mid (a, b) \in \mathbb{Q}^2\}.$$

1. Show that $(\mathbb{Q}[\sqrt{\alpha}], +, \times)$ is a field where $+$ and $\times$ are the usual operations in $\mathbb{R}$.

Let $(\mathbb{F}, +_{\mathbb{F}}, \cdot_{\mathbb{F}})$ and $(\mathbb{K}, +_{\mathbb{K}}, \cdot_{\mathbb{K}})$ be two fields. A *field homomorphism* is a function $\Phi : \mathbb{F} \to \mathbb{K}$ such that for all $a, b \in \mathbb{F}$,

$$\Phi(a +_{\mathbb{F}} b) = \Phi(a) +_{\mathbb{K}} \Phi(b), \quad \Phi(a \cdot_{\mathbb{F}} b) = \Phi(a) \cdot_{\mathbb{K}} \Phi(b),$$
$$\Phi(1_{\mathbb{F}}) = 1_{\mathbb{K}} \quad \text{and} \quad \Phi(0_{\mathbb{F}}) = 0_{\mathbb{K}}.$$

A bijective field homomorphism is called a *field isomorphism.*

2. Let $\alpha, \beta \in \mathbb{N}$ such that $\sqrt{\alpha}$ and $\sqrt{\beta}$ are irrationnal. Give a necessary and sufficient condition on α and β such that there exists a field isomorphism $\Phi : \mathbb{Q}[\sqrt{\alpha}] \to \mathbb{Q}[\sqrt{\beta}]$.

Exercise 4. For $a, b \in \mathbb{R}$, we define

$$a \top b = a + b - 1 \quad \text{and} \quad a * b = ab - a - b + 2.$$

Find a function $\Phi : \mathbb{R} \to \mathbb{R}$ such that Φ is a field isomorphism from $(\mathbb{R}, +, \times)$ to $(\mathbb{R}, \top, *)$ (we do not ask to prove that $(\mathbb{R}, \top, *)$ is a field). What are the identity elements for $\top$ and $*$?

Exercise 5. Let $\mathbb{K}$ be a finite field (a field with a finite number of elements), we denote $\mathbb{K}^* = \mathbb{K} \setminus \{0_{\mathbb{K}}\}$.

1. Construct two examples of finite fields.

2. Compute, for these two examples, $\prod_{x \in \mathbb{K}^*} x$.

3. Compute this quantity for any finite field.

Évariste Galois
(1811–1832)

Leopold Kronecker
(1823–1891)