Homework 8
Math 16310-Section 50

Due: Tuesday June 2nd

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{C}^1$. Compute the derivative of the following functions

1. $g(x, y) = f(y, x)$ **2.** $g(x) = f(x, x)$
3. $g(x, y) = f(y, f(x, x))$ **4.** $g(x) = f(x, f(x, x))$.

Exercise 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

1. Is f continuous on $\mathbb{R}^2$?

2. Is f $\mathcal{C}^1$ on $\mathbb{R}^2$?

3. Is f differentiable on $\mathbb{R}^2$?

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an application $\mathcal{C}^1$ on $\mathbb{R}^2$ and $r \in \mathbb{R}$. We say that f is (positively) homogeneous of degree r if

$$\forall (x, y) \in \mathbb{R}^2,\ \forall t > 0,\ f(tx, ty) = t^r f(x, y).$$

1. Give an example of an homogeneous function of degree 3.

2. Show that if f is homogeneous of degree r then its partial derivatives are homogeneous of degree $r - 1$.

3. Show that if f is homogeneous of degree r then

$$\forall (x, y) \in \mathbb{R}^2,\ x\partial_1 f(x, y) + y\partial_2 f(x, y) = rf(x, y).$$

4. We want to show the other implication, we suppose that f follows the previous identity (called the Euler identity) and we want to show that f is homogeneous of degree r. Consider $\varphi(t) = f(tx, ty)$, by differentiating the function $t \mapsto t^{-r}\varphi(t)$, show that f is homogeneous of degree r.

Exercise 4. Let $p, q \in \mathbb{N}$. We define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^p y^q}{x^2 - xy + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

1. For which p, q is f continuous? Is f differentiable for $p + q = 2$?
Hint : One can first prove that $|xy| \leqslant x^2 - xy + y^2$.

2. We suppose that $p + q = 3$, prove that f is not differentiable at $(0, 0)$.
Hint: One can reason by contradiction.

Exercise 5. *Bonus Exercise.* Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ C^1 such that

$$\partial_1 f(x, y) - 3\partial_2 f(x, y) = 0.$$

Hint: One can use the change of variables $(x, y) \mapsto (ax + by, cx + dy)$.

Carl Gustav Jacob Jacobi
(1804–1851)

Leonhard Euler
(1707–1783)