Homework 7
Math 16310-Section 50

Due: Tuesday May 26th

Exercise 1. Do the following functions have a limit at $\mathbf{0}$ and, if so, what is the limit?

1. $f(x,y,z) = \dfrac{xy+yz}{x^2+2y^2+3z^2}$ **2**. $f(x,y) = \dfrac{1-\cos(xy)}{xy^2}$

3. $f(x,y) = \left(\dfrac{x^2+y^2-1}{x}\sin x, \dfrac{\sin(x^2)+\sin(y^2)}{\sqrt{x^2+y^2}}\right)$ **4**. $f(x,y) = \dfrac{x^\alpha y^\beta}{x^2+y^2}$ for $\alpha, \beta > 0$.

Exercise 2. Let $C \subset \mathbb{R}^2$ be such that for all $x, y \in C$ and $t \in [0,1]$ we have $tx + (1-t)y \in C$ (we say that C is a *convex* subset of $\mathbb{R}^2$) and $f : C \to \mathbb{R}$ be a continuous function.

1. Prove that $f(C)$ is an interval.

2. Let I be an interval of $\mathbb{R}$ and $h : I \to \mathbb{R}$ be a continuous injection. Show that h is strictly monotone by using the function $f(x,y) = h(x) - h(y)$.

Exercise 3. Let A be a non-empty bounded subset of $\mathbb{R}^n$. We say that a function $f : A \to \mathbb{R}^n$ is *Lipschitz* continuous if there exists $k \in \mathbb{R}_+$ such that for all $x, y \in A$, $\|f(x) - f(y)\| \leqslant k\|x - y\|$. We denote $\mathcal{L}$ to be the set of all Lipschitz continuous function from A to $\mathbb{R}^n$.

1. Show that a function in $\mathcal{L}$ is bounded.

2. For $f \in \mathcal{L}$, we define

$$K_f = \left\{k \in \mathbb{R}_+ \mid \forall (x,y) \in A^2,\ \|f(x) - f(y)\| \leqslant k\|x-y\|\right\}.$$

Prove that $c(f) := \inf K_f$ exists and that $c(f) \in K_f$.

3. Let $a \in A$ and $N_a : \mathcal{L} \to \mathbb{R}_+$ be defined by $N_a(f) = c(f) + \|f(a)\|$. Show that $N_a(f)$ follows the same properties as in Theorem 18.10:

- $\forall f \in \mathcal{L}$, $N_a(f) \geqslant 0$ and $N_a(f) = 0 \Rightarrow \forall x \in A, f(x) = 0$
- $\forall f, g \in \mathcal{L}$, $N_a(f+g) \leqslant N_a(f) + N_a(g)$.
- $\forall f \in \mathcal{L}$ and $\lambda \in \mathbb{R}$, $N_a(\lambda f) = |\lambda| N_a(f)$.

4. Let $a, b \in A$, prove that there exists two real numbers $c, C > 0$ such that

$$\forall f \in \mathcal{L}, \quad cN_a(f) \leqslant N_b(f) \leqslant CN_a(f).$$

Exercise 4. *Bonus Exercise.* We define the function $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x) = \frac{1}{\max(1, \|x\|)} x.$$

1. Prove that f is 2-Lipschitz continuous: $\forall x, y \in \mathbb{R}^n, \quad \|f(x) - f(y)\| \leqslant 2\|x - y\|$.

2. Prove that f is 1-Lipschitz continuous:$\forall x, y \in \mathbb{R}^n, \quad \|f(x) - f(y)\| \leqslant \|x - y\|$.

Rudolf Otto Sigismund Lipschitz
(1832–1903)
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Hermann Günther Grassmann
(1809–1877)
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