# Homework 6
# Math 16310-Section 50

Due: Tuesday May 19th

**Exercise 1.** **1.** Let $x, y, z$ be three real numbers such that $2x^2 + y^2 + 5z^2 \leqslant 1$. Show that $(x+y+z)^2 \leqslant \frac{17}{10}$.

**2.** Let $x_1, \dots, x_n \in \mathbb{R}^n$ such that $x_k > 0$ for each $k \in \{1, \dots, n\}$ and $x_1 + \dots + x_n = 1$. Show that $\sum_{k=1}^{n} \frac{1}{x_k} \geqslant n^2$.

**3.** We recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, study whether $\sum_{n=1}^{\infty} u_n$ converges or not with $u_n = \frac{1}{n^2(\sqrt{2})^n} \sum_{k=0}^{n} \sqrt{\binom{n}{k}}$.

**Exercise 2.** Let $x, y, p, q$ be four strictly positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $a_1, \dots, a_n, b_1, \dots, b_n$ be $2n$ strictly positive real numbers.

**1.** Prove that for all $t \in [0,1]$ we have $\ln((1-t)x + ty) \geqslant (1-t)\ln(x) + t\ln(y)$ and prove that

$$xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q.$$

**2.** Prove that $\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$.
*Hint: one can first study the case* $\sum_{i=1}^{n} a_i^p = \sum_{i=1}^{n} b_i^q = 1$.

**3.** We now suppose that $p \geqslant 1$. Prove that $\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}$

**4.** We define $\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$. Show that $\|\cdot\|_p$ follows the properties of Theorem 18.10.

**Exercise 3.** Let $A$ be a bounded non-empty set of $\mathbb{R}^n$. The goal of the exercise is to prove the existence and uniqueness of a closed ball of minimal radius containing $A$.

**1.** Define $D = \{r > 0 \mid A \text{ is contained in a ball of radius } r\}$. Prove that $D$ admits an infimum which we denote $r_0$.

**2.** For $n \geqslant 1$, set $r_n = r_0 + \frac{1}{n}$. Show that there exists $x_n \in \mathbb{R}^n$ such that $A \subset \bar{B}(x_n, r_n)$.

**3.** Show that $(x_n)$ is bounded and deduce the existence of the minimal radius.

**4.** In this exercise, $\|\cdot\|$ denotes the usual Euclidean norm.

**a.** Prove that for all $x, y \in \mathbb{R}^n$,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

**b.** Prove that the ball is unique.

**Challenge** (Not graded): If we define the diameter of $A$ as $d := \sup_{x,y \in A} \|x - y\|$ and $r$ the minimal radius, prove that

$$r \leqslant d\sqrt{\frac{n}{2(n+1)}}.$$

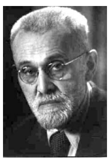
**Exercise 4.** *Bonus Exercise.* Let $u_1, \dots, u_n \in \mathbb{R}^n$ and $C > 0$. We suppose that

$$\forall (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n, \left\| \sum_{i=1}^{n} \varepsilon_i u_i \right\| \leqslant C.$$

Prove that $\sum_{i=1}^{n} \|u_i\|^2 \leqslant C^2$.

Otto Ludwig Hölder
(1859–1937)

Hermann Minkowski
(1864–1909)



Heinrich Wilhelm Ewald Jung
(1876–1953)