Homework 4
Math 16310-Section 50

Due: Tuesday May 4th

Exercise 1. Prove, according to the possible values of $\alpha, \beta \in \mathbb{R}$, whether the series given by (u_n) converges or not, where

$$u_n = \frac{1}{n^\alpha (\ln n)^\beta}$$

Hint: One can use the series–integral comparison method.

Exercise 2. Let (u_n) be a sequence of positive numbers such that there exists $a \in \mathbb{R}$ and $\varepsilon(n) \xrightarrow[n\to\infty]{} 0$ such that

$$\frac{u_{n+1}}{u_n} = 1 - \frac{a}{n} + \frac{\varepsilon(n)}{n}.$$

1. Show, by comparison, that $\sum_n u_n$ converges if $a > 1$.

2. Show that $\sum_n u_n$ diverges if $a < 1$.

3. By using the previous exercise, show that things are not that simple for $a = 1$.

4. We suppose that we can write

$$\frac{u_{n+1}}{u_n} = 1 - \frac{1}{n} + \frac{a(n)}{n^2}$$

where $a(n)$ is a bounded sequence. We denote $v_n = \ln(nu_n)$ and $w_n = v_{n+1} - v_n$.

a. Show that for n large enough we have $|w_n| \leqslant \frac{C}{n^2}$ for some $C > 0$.

b. Prove that $nu_n \xrightarrow[n\to\infty]{} \lambda$ for some $\lambda > 0$ and that $\sum_n u_n$ is divergent.

Exercise 3. Let (u_n) be a sequence of real numbers such that $\sum_n u_n$ is convergent but $\sum_n |u_n|$ is divergent.

1. Give an example of such a sequence.

2. We denote $A = \{n \in \mathbb{N} \mid u_n \geqslant 0\}$ and $B = \mathbb{N} \setminus A$. Prove that A and B are infinite.

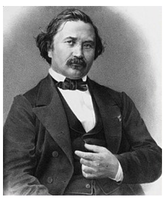
3. Prove that $\sum_{n\in A} u_n$ and $\sum_{n\in B} u_n$ are divergent.

4. Let $\alpha \in \mathbb{R}$, construct a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} u_{\sigma(n)} \xrightarrow[n\to\infty]{} \alpha$.

Hint: Construct σ inductively and then check that it is a bijection. After constructing σ, check that the sum does converge to α.

5. Give the start of the construction of the bijection σ for the example you gave in **Question 1.** with your choice of $\alpha \in \mathbb{R}$.

Exercise 4. *Bonus Exercise.* Let A be the set of positive integers such that there is no 9 in their base 10 digit representation. We enumerate A by an increasing sequence (k_n). Does $\sum_n \frac{1}{k_n}$ converges?



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Joseph Ludwig Raabe
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Jean-Marie Constant Duhamel
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