Homework 2
Math 16310-Section 50

Due: Tuesday April 28th

Exercise 1. Let $(p_k)_{k\geqslant 1}$ be the ordered sequence of prime numbers. The goal of the exercise is to find the nature of the series $\sum_{k\geqslant 1}\frac{1}{p_k}$. For $n \geqslant 1$, we denote $V_n = \prod_{k=1}^{n}\frac{1}{1-\frac{1}{p_k}}$.

1. Show that the sequence (V_n) is convergent if, and only if, the sequence $(\ln V_n)$ is convergent.

2. Show that that the sequence (V_n) is convergent if, and only if, the series $\sum_{k\geqslant 1}\frac{1}{p_k}$ is convergent.

3. Prove that

$$V_n = \prod_{k=1}^{n}\left(\sum_{j\geqslant 0}\frac{1}{p_k^j}\right).$$

and that $V_n \geqslant \sum_{j=1}^{n}\frac{1}{j}$.

4. What is the nature of the series $\sum_{k\geqslant 1}\frac{1}{p_k}$?

5. For $\alpha\in\mathbb{R}$, what is the nature of the series $\sum_{k\geqslant 1}\frac{1}{p_k^\alpha}$.

Exercise 2. We consider two sequence of real numbers (u_n) and (v_n). We are interested in the convergence of the series $\sum_{n\geqslant 0}u_n v_n$. For $n\geqslant 0$, we denote $S_n=\sum_{k=0}^{n}u_k$.

1. Show that for all $(p,q)\in\mathbb{N}^2$ such that $p\leqslant q$ we have

$$\sum_{k=p}^{q}u_k v_k = S_q v_q - S_{p-1}v_p + \sum_{k=p}^{q-1}S_k(v_k - v_{k+1}).$$

2. Show that if the sequence (S_n) is bounded, if $v_n\in\mathbb{R}_+$ for all $n\geqslant 0$ and if (v_n) is decreasing such that $v_n \xrightarrow[n\to\infty]{} 0$ then the series $\sum_{n\geqslant 0}u_n v_n$ is convergent.

3. Show that the series $\sum_{n\geqslant 1}\frac{\sin(n\theta)}{\sqrt{n}}$ converges for all $\theta\in\mathbb{R}$.
Hint: Recall that $\Im(\mathrm{e}^{\mathrm{i}n\theta})=\sin(n\theta)$.

Exercise 3. The goal of this exercise is to compute $\sum_{n\geqslant 1}\frac{1}{n^2}$.

1. Let f be a C^1 function on $[0,\pi]$ (the function is continuous, differentiable and its derivative is continuous). Show that

$$\int_0^\pi f(t)\sin\left(\frac{(2n+1)t}{2}\right)\mathrm{d}t \xrightarrow[n\to\infty]{} 0.$$

2. We denote $A_n(t)=\frac{1}{2}+\sum_{k=1}^{n}\cos(kt)$. Prove that for all $t\in(0,\pi]$ we have

$$A_n(t)=\frac{\sin((2n+1)t/2}{2\sin(t/2)}.$$

Hint: Recall that $\Re(\mathrm{e}^{\mathrm{i}kt})=\cos(kt)$.

3. Find two real numbers a and b such that for all $n\geqslant 1$,

$$\int_0^\pi (at^2+bt)\cos(nt)\mathrm{d}t = \frac{1}{n^2}.$$

and then see that

$$\int_0^\pi (at^2+bt)A_n(t) = \sum_{k=1}^{n}\frac{1}{k^2}-\frac{\pi^2}{6}.$$

4. Using **Question 1.**, prove that $\sum_{k=1}^{n} \frac{1}{k^2} \xrightarrow[n\to\infty]{} \frac{\pi^2}{6}$.

Exercise 4. *Bonus exercise.* Let (a_n) be a sequence of positive real numbers such that $\sum_{n\geqslant 1} a_n$ converges.

1. Prove that

$$\sum_{n\geqslant 1} \frac{a_1 + 2a_2 + \cdots + na_n}{n(n+1)} \text{ converges and that } \sum_{n\geqslant 1} \frac{a_1 + 2a_2 + \cdots + na_n}{n(n+1)} = \sum_{n\geqslant 1} a_n.$$

2. Show that

$$\frac{1}{(n!)^{1/n}} \leqslant \frac{\mathrm{e}}{n+1}.$$

3. Prove that

$$\sum_{n=1}^{\infty} (a_1 \dots a_n)^{1/n} \leqslant \mathrm{e} \sum_{n=1}^{\infty} a_n.$$

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