Homework 2
Math 16310-Section 50

Due: Tuesday April 21st

Exercise 1. Let (u_n) and (v_n) be the two sequences defined by for $n \geqslant 1$,

$$u_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad v_n = u_n + \frac{1}{n \times n!}.$$

1. Show that (u_n) is increasing, (v_n) is decreasing and that $|u_n - v_n| \xrightarrow[n\to\infty]{} 0$.

2. Prove that (u_n) and (v_n) both converge to the same limit. We denote e their common limit.

3. Prove that for all $n \in \mathbb{N}$, $n!u_n < n!\mathrm{e} < n!u_n + \frac{1}{n}$.

4. Prove that e is an irrational number.

Exercise 2. *Add. Ex. 14 from Script 15.* A *Cauchy sequence* (of rational numbers) is a sequence (a_n) with $a_n \in \mathbb{Q}$ such that, given an $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|a_n - a_m| < \epsilon$.

1. Prove that any sequence of rationals that converges in $\mathbb{Q}$ is a Cauchy sequence.

2. Prove that any Cauchy sequences are bounded.

3. Prove that if (x_n) and (y_n) are Cauchy sequences then so are $(x_n + y_n)$ and$(x_n y_n)$.

4. Prove that the set of Cauchy sequences of rationals has additive and multiplicative identities, additive inverses, but many Cauchy sequences fail to have multiplicative inverses.

5. Define X to be the set of all Cauchy sequences of rational numbers. Then we define an equivalence relation on X by $(a_n) \sim (b_n)$ if $\lim_{n\longrightarrow\infty} |a_n - b_n| = 0$. Prove that this is indeed an equivalence relation. We let $\mathcal{R}$ denote the set of equivalences classes of Cauchy sequences of rationals and define addition and multiplication in $\mathcal{R}$ by

$$\begin{aligned} [(a_n)] + [(b_n)] &= [(a_n + b_n)] \\ [(a_n)][(b_n)] &= [(a_n b_n)]. \end{aligned}$$

Note that we can view $\mathbb{Q}$ as a subset of $\mathcal{R}$ by identifying a rational number q with the equivalence class of the constant sequence, $[(q, q, q, ...)]$.

6. Prove that $\mathcal{R}$ forms a field that satisfies Axioms 1-3. *Note: The field axioms should all be very routine, except for FA8.*

7. We say that an equivalence class $[(a_n)]$ is *positive* and write $[(a_n)] > 0$,if there is some $N \in \mathbb{N}$ and $\epsilon \in \mathbb{Q}, \epsilon > 0$, such that $a_n > \epsilon$ for all $n \geq N$. We say that $[(a_n)] > [(b_n)]$ if $[(a_n - b_n)] > 0$. Prove that $<$ is well-defined and that $\mathcal{R}$ is an ordered field with this ordering.

8. Does every nonempty bounded subset of $\mathcal{R}$ have a supremum?
Hint : Let X be a bounded subset of $\mathcal{R}$. Consider

$$Y = \{q \in \mathbb{Q} \mid (q, q, q, \cdots)] \textit{ is not an upper bounded of } X\}.$$

Construct a suitable sequence (q_n) of points in Y. Lemma 6.10 should be helpful.

9. Does $\mathcal{R}$ satisfy Axiom 4?

10. Does $\mathcal{R}$ satisfy Axiom 5?

11. What can you say about $\mathcal{R}$?

Exercise 3. *Bonus exercise.* Let f be the function defined in $\mathbb{R}$ by $f(x) = x - x^2$ and (u_n) be the sequence defined by $u_0 \in (0,1)$ and $u_{n+1} = f(u_n)$.

1. Study the function f.

2. Show that for all $n \geqslant 0$, $0 < u_n < \frac{1}{n+1}$.

3. Prove that the sequence (v_n) defined by $v_n = nu_n$ for $n \geqslant 0$ is increasing.

4. Prove that (v_n) converges to some limit $\ell \in (0,1]$ (we do not ask to find ℓ yet).

5. Define $w_n = n(v_{n+1} - v_n)$. Show that (w_n) converges to $\ell(1-\ell)$.

6. Let (t_n) be another sequence such that there exists $n_0 \geqslant 1$ such that for all $n \geqslant n_0$ we have

$$t_{n+1} - t_n \geqslant \frac{a}{n}$$

for some $a > 0$. Show that $t_{2n} - t_n \geqslant \frac{a}{2}$ and that (t_n) diverges.

7. Show that if $\ell \neq 1$, the sequence (v_n) follows the same inequality that (t_n) in the previous question. Find the value of ℓ.

Jean-Baptiste Joseph Fourier
(1768–1830)

Hugues Charles Robert Méray
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