Homework 1
Math 16310-Section 50

Due: Tuesday April 14th

Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be $C^2([a, b])$. We define $I := \int_a^b f(t)\mathrm{d}t$ and $I_{\text{med}} := (b-a)f(\frac{a+b}{2})$. Denote $M_2 := \max\{|f''(x)| \mid x \in [a, b]\}$.

1. Let $\Delta(x) := \int_{c-x}^{c+x} f(t)\mathrm{d}t - 2xf(c)$ where $c = \frac{a+b}{2}$. Show that for all $x \in [0, \frac{b-a}{2}]$, $|\Delta''(x)| \leqslant 2xM_2$.

2. Prove that $|I - I_{\text{med}}| \leqslant M_2 \frac{(b-a)^3}{24}$.

3. For all $n \geqslant 1$, denote $I_{\text{med},n} = \frac{b-a}{n} \sum_{k=0}^{n-1} f(\frac{x_k + x_{k+1}}{2})$ with $x_k = a + k\frac{b-a}{n}$. Show that

$$\left| \int_a^b f(t)\mathrm{d}t - I_{\text{med},n} \right| \leqslant \frac{(b-a)^3}{24n^2} M_2.$$

Note that the convergence is faster than in Exercise 7 of last homework since $\frac{1}{n^2}$ goes faster to zero than $\frac{1}{n}$.

Exercise 2. Show that the following sequences converge or not. If so, give the limit.

$$\textbf{1.} \quad u_n = \left(2\sin\left(\frac{1}{n}\right) + \frac{3}{4}\cos(n)\right)^n. \quad \textbf{2.} \quad v_n = \cos\left(\left(n + \frac{1}{n}\right)\pi\right).$$

3. $w_n = \sin((3+\sqrt{5})^n \pi)$. *Hint: one can first show that* $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ *is an even integer.*

Exercise 3. Let $(u_n)_{n \geqslant 1}$ be a sequence of real numbers. We denote $S_n = \frac{u_1 + \cdots + u_n}{n}$.

1. Show that if (u_n) converges to 0 then (S_n) converges to 0.

2. Show that if (u_n) converges to $\ell \in \mathbb{R} \cup \{+\infty, -\infty\}$ then (S_n) converges to ℓ.

3. Is the converse true?

Exercise 4. For $n \in \mathbb{N} \cup \{0\}$ we define $W_n = \int_0^{\frac{\pi}{2}} \sin^n(x)\mathrm{d}x$.

1. Show that for all $n \in \mathbb{N} \cup \{0\}$, $W_n = \int_0^{\frac{\pi}{2}} \cos^n(x)\mathrm{d}x$.

2. Show that (W_n) is decreasing.

3. Show that for all $n \in \mathbb{N} \cup \{0\}$ we have $(n+2)W_{n+2} = (n+1)W_n$. and

4. Show that for all $p \in \mathbb{N} \cup \{0\}$,

$$W_{2p} = \frac{(2p)!\pi}{2^{2p+1}(p!)^2} \quad \text{and} \quad W_{2p+1} = \frac{2^{2p}(p!)^2}{(2p+1)!}.$$

5. Show that $(n+1)W_{n+1}W_n = \frac{\pi}{2}$. and that $\frac{W_{n+1}}{W_n} \xrightarrow[n\to\infty]{} 1$.

6. Show that $\sqrt{\frac{2n}{\pi}} W_n \xrightarrow[n\to\infty]{} 1$ we also write $W_n \sim_{+\infty} \sqrt{\frac{\pi}{2n}}$.

Exercise 5. *Bonus exercise.* Let $(u_n)_{n \geqslant 0}$ and $(v_n)_{n \geqslant 0}$ be two sequences converging respectively to u and v. Show that $(\frac{1}{n+1} \sum_{k=0}^n u_k v_{n-k})$ converges to uv.

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(1859–1906)

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