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MATH 20400 Section 51
Analysis in $\mathbb{R}^n$

Midterm 2

February 28th 2020

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Directions: You have 50 minutes for this midterm. No calculators, notes, books, laptops, phones, etc. are allowed. If you see an error in the exam, correct it in your solution and explain why you made the correction. No questions will be answered during the exam. Every answer has to be justified and proofs must be complete.

Initials:____

Exercise 1. **1.** State if the following statements are true or false. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

Statement	**True**	**False**
If $\partial_{ij} f(x) = \partial_{ji} f(x)$ for all $x \in \mathbb{R}^n$ then $f \in C^2(\mathbb{R}^n)$.		
If $f \in C^k(\mathbb{R}^n)$, $\ell \in \{1, \dots, k\}$ and $i_1, \dots, i_\ell \in \{1, \dots n\}$ then for any permutation $\sigma \in \mathfrak{S}_\ell$, $\partial_{i_1 \dots i_\ell} f = \partial_{i_{\sigma(1)} \dots i_{\sigma(\ell)}} f$.		
If $n > 1$ and if f is differentiable at $x \in \mathbb{R}^n$ with $\mathrm{d}f_x = 0$ then x is a local extremum of f.		
If $f \in C^2(\mathbb{R}^n)$, if $\nabla f(x) = 0$ and $\det \operatorname{Hess}_f(x) < 0$ then x is a saddle point of f.		
If $f \in C^1(\mathbb{R}^n)$ and if $\mathscr{S} = \{x \in \mathbb{R}^n \mid g(x) = 0\}$ with $g \in C^1(\mathbb{R}^n)$. If $f_{\mid \mathscr{S}}$ has a local extremum at a and if $\nabla g(a) \neq 0$ then $\exists \lambda \in \mathbb{R}, \nabla f(a) = \lambda \nabla g(a)$.		

2. Choose a statement, if there is one, you declared **False** and give a counter-example (prove that it is a counter-example).

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Exercise 2. We consider the surface $\mathscr{S} = \{(x, y, z) \in \mathbb{R}^3 \mid z = 4x^2 + y^2\}$.

1. Prove that $\mathscr{S}$ is a smooth hypersurface of $\mathbb{R}^3$.

2. Find the points on $\mathscr{S}$ where the tangent plane is parallel to the plane of equation $x+2y+z = 6$.

3. Same question with the plane $3x + 5y - 2z = 3$.

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Exercise 3. Find the local and global extrema of the following functions

1. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x, y) = -2(x - y)^2 + x^4 + y^4.$$

2. $f : \mathbb{R}^3 \to \mathbb{R}$ defined by
$$f(x, y, z) = yz - x^2 - y^2 - z^2.$$

Initials:____

Exercise 4. Find the global extrema of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^3 + y^3$ under the constraint $x^2 + y^2 = 1$.

Initials:____

Exercise 5 $(*)$. **1.** Find the extrema of $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \|x\|^2$ on the subset of $\mathbb{R}^n$ $C = \{x \in \mathbb{R}^n \mid \langle Ax, x\rangle = 1\}$ where A is a real symmetric matrix and $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ is the dot product.

2. Apply your result to $n = 2$ and

$$A = \begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$$

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