Last Name: ________________________________

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MATH 20400 Section 51
Analysis in $\mathbb{R}^n$

Midterm 1

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Directions: You have 50 minutes for this midterm. No calculators, notes, books, laptops, phones, etc. are allowed. If you see an error in the exam, correct it in your solution and explain why you made the correction. No questions will be answered during the exam. Every answer has to be justified and proofs must be complete.

Initials:____

Exercise 1. **1.** State if the following statements are true or false. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $\mathbf{e} = (e_1, \dots, e_n)$ be denoting the standard basis.

Statement	**True**	**False**
If f admits a directional derivative in any direction at $x \in \mathbb{R}^n$ then f is continuous at x.		
If the Jacobian matrix of f, Jac_f, exists on $\mathbb{R}^n$ then f is differentiable on $\mathbb{R}^n$ and $\mathrm{Mat}_{\mathbf{e}}(\mathrm{d}f_x) = \mathrm{Jac}_f(x)$.		
If f is $C^1(\mathbb{R}^n)$ then f is differentiable on $\mathbb{R}^n$.		
If f is differentiable on $\mathbb{R}^n$ then f is $C^1(\mathbb{R}^n)$.		
If f is a linear map then f is differentiable.		

2. Choose a statement, if there is one, you declared **False** and give a counter-example (prove that it is a counter-example).

Initials:____

Exercise 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \cos(x^2 - y^2)$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $g(x, y) = (x - y, x + y)$.

1. Justify that f and g are differentiable on $\mathbb{R}^2$ and compute the Jacobian matrix of f and g for any $(x, y) \in \mathbb{R}^2$.

2. Let $(x, y) \in \mathbb{R}^2$, for any $(h_1, h_2) \in \mathbb{R}^2$ find $\mathrm{d}(f \circ g)_{(x,y)}(h_1, h_2)$ using the two following methods:

a. By using the Jacobian matrices and the chain rule.

b. By computing $f \circ g$ directly.

Initials:____

Exercise 3. We define $f_a : \mathbb{R}^2 \to \mathbb{R}$ by

$$f_a(x,y) = \begin{cases} \dfrac{y^2}{(x^2+y^2)^{7/9}} & \text{if } (x,y) \neq (0,0), \\ \\ a & \text{otherwise.} \end{cases}$$

1. Find the value of a so that f_a is continuous on $\mathbb{R}^2$.

2. For this value of a, do $\partial_1 f_a$ and $\partial_2 f_a$ exist at $(0,0)$?

Initials:____

Exercise 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $C^1(\mathbb{R}^2)$.

1. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(3t + 2t^2, t^3)$. Show that g is $C^1(\mathbb{R})$ and compute $g'(t)$ in terms of the partial derivatives of f.

2. We define $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(x, y) = f(xy^2, xy + y^2)$. Show that h is $C^1(\mathbb{R}^2)$ and compute ∇h in terms of the partial derivatives of f.

Initials:____

Exercise 5 $(*)$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1(\mathbb{R}^n)$.

1. Let $x \in \mathbb{R}^n$. Show that

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \|x - y\| \leqslant \delta \Rightarrow |||\mathrm{d}f_x - \mathrm{d}f_y||| \leqslant \varepsilon.$$

2. Show that if $\mathrm{d}f_0 = \mathrm{Id}_{\mathbb{R}^n}$ then there exists a neighborhood V of 0 such that the restriction of f to V is injective.
Hint: Use the function $g(x) = f(x) - x$.

Initials:____