Homework 1
Math 16110-Section 50

Due: Tuesday October 8th

Exercise 1. *Don't worry, the proof is false.* We are going to show by induction that every MATH-16110 student will have the same GPA. We define the following proposition

$$P(n) = \{n \text{ students will have the same GPA}\}.$$

- Firstly, $P(1)$ is true since we have only one student.
- Suppose now that $P(n)$ is true and consider $n+1$ students written $s_1, \dots, s_{n+1}$.
  By the induction hypothesis: $s_2, \dots, s_{n+1}$ will have the same GPA
  $s_1, \dots, s_n$ will have the same GPA.
  In particular, s_1 and s_{n+1} will have the same GPA. We can then conclude that $s_1, \dots, s_{n+1}$ will have the same GPA or, in other words, $P(n+1)$ is true.

1. Explain what is wrong in this proof.

Exercise 2. *A harder induction.* Consider the partial harmonic sum

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

1. Show that for $n \in \mathbb{N}$ we can write

$$H_n = \frac{p_n}{q_n} \quad \text{with } p_n \text{ an odd integer and } q_n \text{ an even integer.}$$

Exercise 3. *On set equations.* Let A, $B \subset E$. Solve the two following equations of unknown $X \subset E$

1. $A \cup X = B$.

2. $A \cap X = B$.

Exercise 4. *On the powerset.* Let E be a finite set, we denote $\mathcal{P}(E)$ the set of subsets of E,

$$\mathcal{P}(E) = \{A,\ A \subset E\}.$$

1. Write out the elements of $\mathcal{P}(\{a,b\})$ and $\mathcal{P}(\{a,b,c\})$.

2. Show by induction that for E a set with n elements, the number of elements of $\mathcal{P}(E)$ is 2^n.

3. Give the number of elements of $\mathcal{P}(\mathcal{P}(\{a,b,c\}))$.

4. We are interested in the number of subsets $\mathcal{T}$ of $\mathcal{P}(\{a,b,c\})$ with the following properties:

(i) $\emptyset \in \mathcal{T}$ and $\{a,b,c\} \in \mathcal{T}$

(ii) For all A, $B \in \mathcal{T}$, $A \cap B \in \mathcal{T}$.

(iii) For all A, $B \in \mathcal{T}$, $A \cup B \in \mathcal{T}$.

Give the total number of distinct subsets $\mathcal{T}$ of $\mathcal{P}(\{a,b,c\})$ verifying the three properties above.

Exercise 5. *On the Zermelo–Fraenkel definition of* $\mathbb{N}$. We want to give a set theoretic definition of the natural numbers $\mathbb{N}$. As seen in Script 0, we need to give a "special element" 1 and a "successor function" $S : \mathbb{N} \to \mathbb{N}$,

$$1 = \emptyset \in \mathbb{N}, \quad S(n) = n \cup \{n\} \quad \text{for } n \in \mathbb{N}.$$

1. Write $2 = S(1)$, $3 = S(S(1))$ and $4 = S(S(S(1)))$.

2. We suppose that Peano axioms I, II, III and V hold. The goal of this question is to show Axiom IV.

a. Show that if $n \in \mathbb{N}$ and $m \in n$ then $m \subseteq n$.

b. Show that if $n \in \mathbb{N}$ and $m \in n$ then $n \not\subseteq m$.

c. Show Axiom IV.

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