Homework 8
Math 16110-Section 50

Due: Tuesday November 26th

Exercise 1. Let A and B be two nonempty bounded subsets of $\mathbb{R}$ and $x \in \mathbb{R}$. We denote

$$-A = \{-a \mid a \in A\}, \quad A + B = \{a + b \mid a \in A,\, b \in B\},$$
$$x + A = \{x + a \mid a \in A\}, \quad \text{and} \quad AB = \{ab \mid a \in A,\, b \in B\}$$

1. Show that $\sup(-A) = -\inf(A)$.

2. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

3. Show that $\sup(x + A) = x + \sup(A)$.

4. Do we have $\sup(AB) = \sup(A)\sup(B)$? What assumptions can we add for this equality to be true?

Exercise 2. **1.** Show that for all real numbers x_i, $x_j > 0$ we have

$$\frac{x_i}{x_j} + \frac{x_j}{x_i} = \frac{x_i^2 + x_j^2}{x_i x_j} \geqslant 2.$$

2. Let $n \in \mathbb{N}$. Find

$$\inf\left\{(x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \,\middle|\, x_1, \ldots, x_n > 0\right\}$$

Exercise 3. *On the Heine-Borel property*. An *open cover* of a set X is a collection of open sets whose union contains X as a subset. In other words,

$$C = \{U_\alpha \mid \alpha \in A\} \text{ is an } \textit{open cover} \text{ of } X \iff X \subseteq \bigcup_{\alpha \in A} U_\alpha \text{ and } U_\alpha\text{'s are open sets.}$$

We call a cover a finite cover is there is a finite number of open sets U_α.

In $\mathbb{R}$, we call a set S *compact* if we can extract a finite subcover from any open cover of S: If there exists $C = \{U_\alpha \mid \alpha \in A\}$ such that $S \subseteq \bigcup_{\alpha \in A} U_\alpha$ with U_α open sets then there exist $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in A$ such that $S \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

The goal of the exercise is to prove that a segment $[a, b] = \{a\} \cup \underline{ab} \cup \{b\} \subset \mathbb{R}$ is compact.

1. Let $[a, b]$ be a segment and $C = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $[a, b]$.

Show that $m = \sup\{x \in [a, b] \mid [a, x] \text{ is covered by a finite number of open sets } U_j\}$ exists.

2. Show that there exists $i \in A$ such that $m \in U_i$.

3. Finish by proving that $m \geqslant b$.

Exercise 4. *Countable partition of* $[0, 1]$. The goal of the exercise is to prove that $[0, 1]$ is not the countable disjoint union of nonempty closed sets. Suppose that we can and denote

$$[0, 1] = \bigcup_{n \geqslant 0} F_n \text{ where } F_n \text{ are nonempty pairwise disjoint closed sets.}$$

For **Question 2.** of this exercise, you need to remember **Exercise 1.** from **Homework 6.** and see that every open set of $[0, 1]$ can be written as a union of disjoint regions (even countable union). Each of these open intervals is called a *connected component* of the open set.

1. Prove that if we can construct a sequence of regions (I_n) of $[0,1]$ such that

(i) $I_n \subset \overline{I_n} \subset I_{n-1}$,

(ii) $I_n \cap F_n = \emptyset$

we obtain a contradiction.

2. We can suppose without loss of generality that 0 and 1 are in F_0.

a. Prove that $[0,1] \setminus F_0$ is a nonempty open set and define I_0 to be one of its connected component.

b. Define k_1 to be the smallest $k \in \mathbb{N}$ such that $I_0 \cap F_k \neq \emptyset$. Prove that $I_0 \setminus F_{k_1}$ is a nonempty open set and prove that we can construct a region $I_1 \subset I_0 \setminus F_{k_1}$ such that $\overline{I_1} \subset I_0$.

c. Finish the construction of the sequence $(I_n)_{n \geqslant 0}$ by induction and obtain the result.

Eduard Heine
(1821–1881)

Émile Borel
(1871–1956)

Fun fact: É. Borel is my academic great-great-great-grandfather:

É. Borel → G. Valiron → L. Schwartz → A. S. Üstünel → P. Bourgade → L. Benigni