Homework 7
Math 16110-Section 50

Due: Tuesday November 19th

Exercise 1. *On Fréchet's definition of continuity.* Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two continua and $f : \mathcal{C}_1 \to C_2$. We say that f is *continuous* if, and only if,

$$\text{For all open sets } V \subset \mathcal{C}_2,\ f^{-1}(V) \subset \mathcal{C}_1 \text{ is open.}$$

1. Prove that f is continuous if, and only if, for all closed sets $F \subset \mathcal{C}_2$, $f^{-1}(F) \subset \mathcal{C}_1$ is closed.

2. Prove that f is continuous if, and only if, for all regions $R \subset \mathcal{C}_1$, $f^{-1}(R) \subset \mathcal{C}_1$ is open.

3. Prove that f is continuous if, and only if, for all $a \in \mathcal{C}_2$ the sets $\{x \in \mathcal{C}_1 \mid f(x) < a\}$ and $\{x \in \mathcal{C}_1 \mid f(x) > a\}$ are open.

4. Prove that f is continuous if, and only if, for all $A \subset \mathcal{C}_1$, $f(\bar{A}) \subset \overline{f(A)}$.

5. Prove that f is continuous if, and only if, for all $B \subset \mathcal{C}_2$, $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.

6. Prove that f is continuous if, and only if, for all $C \in \mathcal{C}_2$, $\mathrm{Fr}\left(f^{-1}(C)\right) \subset f^{-1}(\mathrm{Fr}(C))$.

7. Find examples where the last three inclusions are strict.

Exercise 2. *On Cauchy's definition of continuity.* We recall the definition of the absolute value of $x \in \mathbb{R}$,

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

1. Prove that for a region $R = \underline{ab} \subset \mathbb{R}$ we have

$$R = \{x \in \mathbb{R} \mid a < x < b\} = \left\{ x \in \mathbb{R} : \left| x - \frac{a+b}{2} \right| < \frac{b-a}{2} \right\}.$$

2. Prove that $f : \mathbb{R} \to \mathbb{R}$ is continuous if, and only if, for all $a \in \mathbb{R}$,

$$\text{for every } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that for all } x \in \mathbb{R},\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

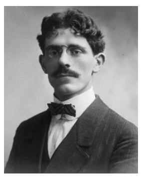
Exercise 3. *On coverings of segments.* For a region $R = \underline{ab} \subset \mathbb{R}$, we denote $\mu(R) = b - a$ its length. We suppose that for every real number x we give a region R_x such that $\mu(R_x) \leqslant c$ for some fixed $c > 0$ (independent of x). We also denote $[a, b] = \underline{ab} \cup \{a\} \cup \{b\}$. We suppose that we can find a finite number m of real numbers y_i such that

$$[a, b] \subset \bigcup_{i=1}^{m} R_{y_i} \quad \text{(this will probably be proved in a future homework).}$$

Prove that we can find a finite number of real numbers $x_1, \ldots, x_n$ such that

$$[a, b] \subset \bigcup_{i=1}^{n} R_{x_i} \quad \text{and} \quad \sum_{i=1}^{n} \mu(R_{x_i}) \leqslant 2(b - a) + c.$$

Augustin Louis Cauchy
(1789–1857)



René Maurice Fréchet
(1878–1973)

Fun fact: R. M. Fréchet is my academic great-great-great-great-grandfather:

R. M. Fréchet → R. Fortet → J. Neveu → P. Priouret → M. Yor → P. Bourgade → L. Benigni