Homework 6
Math 16110-Section 50

Due: Tuesday November 12th

Exercise 1. Let U be an open set of a continuum C. We define a relation on the elements of U by

$$x\mathcal{R}y \iff \begin{cases} \underline{xy} \subset U & \text{if } x \leqslant y \\ \underline{yx} \subset U & \text{if } x > y \end{cases}$$

1. Show that $\mathcal{R}$ is an equivalence relation. We denote $[x]$ the equivalence class of x.

2. Show that if $a, b \in [x]$ with $a < b$ then we have for every $y \in C$, $y \in \underline{ab} \Rightarrow y \in [x]$.

3. Show that $[x]$ is open.

4. Show that U can be written as the union of disjoint open sets following the property from **Question 2.**

Exercise 2. Let $A \subset C$ with C a continuum. We say that $x \in A$ is an *interior* point of A if there is a region R such that $x \in R \subset A$. We let $\mathring{A} = \{a \in A \mid a \text{ is an interior point of } A\}$.

1. Let $A = \underline{ab}$, for $a, b \in C$ and $a < b$. Find $\mathring{A}$.

2. Show that A is open if, and only if, $A = \mathring{A}$.

3. Show that $\mathring{A}$ is open.

4. Let A, B two subsets of C.

- **a.** Show that if $A \subset B$ then $\mathring{A} \subset \mathring{B}$ and $\bar{A} \subset \bar{B}$.
- **b.** Show that $(A \cap B)^{\mathrm{o}} = \mathring{A} \cap \mathring{B}$ and that $\mathring{A} \cup \mathring{B} \subset (A \cup B)^{\mathrm{o}}$. Is the other inclusion also true?
- **c.** Compare $\overline{A \cap B}$ and $\bar{A} \cap \bar{B}$, then $\overline{A \cup B}$ and $\bar{A} \cup \bar{B}$.

Exercise 3. Let C be a continuum.

1. Show that if A, B are subsets of C such that $A \subset B$ then, if B is closed, $\bar{A} \subset B$.

2. Show that if A, B are subsets of C such that $A \subset B$ then, if A is open, $A \subset \mathring{B}$

3. Let $A \subset C$ and $\mathscr{F}_A = \{B \subset C \mid B \text{ is a closed set containing } A\}$. Show that $\overline{A} = \bigcap_{B \in \mathscr{F}_A} B$.

4. Formulate and prove a result analogous to the previous question for $\mathring{A}$.

Exercise 4. Let A be a subset of a continuum C. We define the frontier of A as $\mathrm{Fr}(A) = \bar{A} \setminus \mathring{A} = \bar{A} \cap \overline{E \setminus A}$.

1. Show that $\mathrm{Fr}(A) = \mathrm{Fr}(C \setminus A)$.

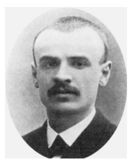
2. Show that A is closed if, and only if, $\mathrm{Fr}(A) \subset A$.

3. Show that A is open if, and only if, $\mathrm{Fr}(A) \cap A = \emptyset$.

4. Show that if A is closed then $\mathrm{Fr}(\mathrm{Fr}(A)) = \mathrm{Fr}(A)$.

Exercise 5. Find an example of a continuum C and a subset A such that A, $\bar{A}$, $\mathring{A}$, $\bar{\mathring{A}}$, $\mathring{\bar{A}}$ are pairwise distinct.

Karl Weierstrass
(1815–1897)



René-Louis Baire
(1874–1932)

K. Weierstrass introduced the notion of limit points while R.-L. Baire first gave the definition of an open set. Note that the actual term “open” was coined later by H.-L. Lebesgue.