Homework 5
Math 16110-Section 50

Due: Tuesday November 5th

Exercise 1. *An ordering of the integers.* We define a relation on $\mathbb{N}$:

$$p \leqslant_{\mathbb{N}} q \iff \text{There exists } k \in \mathbb{N},\, q = p^k.$$

1. Show that $(\mathbb{N}, \leqslant_{\mathbb{N}})$ is an ordered set in the sense of Homework 4. Is it a totally ordered set?

2. Find the upper bounds of the subset $\{2, 4\}$. Same question for $\{2, 3\}$.

Exercise 2. *On Gaussian integers.* Let $\mathrm{i}^2 = -1$ and define $\mathbb{Z}[\mathrm{i}] = \{a + \mathrm{i}b \mid a, b \in \mathbb{Z}\}$. We define a relation on $\mathbb{Z}[\mathrm{i}]$ by

$$a + \mathrm{i}b < c + \mathrm{i}d \iff a < c \text{ or } (a = c \text{ and } b < d)\,.$$

1. Prove that $<$ is an ordering in the sense of the script (*trichotomy + transitivity*).

2. Prove that $(\mathbb{Z}[\mathrm{i}], <)$ satisfies Axioms 1,2, and 3 of the script.

Exercise 3. We define a relation on $\mathbb{R}^2$ by

$$(x, y) \leqslant (x', y') \iff x \leqslant x' \text{ and } y \leqslant y'.$$

1. Show that $(\mathbb{R}^2, \leqslant)$ is an ordered set in the sense of Homework 4.

2. Give all the upper bounds of

$$\mathbb{D} = \left\{(x, y) \mid x^2 + y^2 = 1\right\}$$

Exercise 4. Let $(E, \leqslant)$ be an ordered set as in Homework 4 and $A, B \subset E$. We suppose that A and B both have a maximal element.

1. Does $A \cup B$ have a maximal element?
Hint: Consider the cases where the order is total or not.

2. Does $A \cap B$ have a maximal element?

Exercise 5. *On the well-ordering theorem.* In this exercise, we admit this version of Zorn's lemma (actually weaker than the version you proved in Homework 4). Let $(X, \leqslant)$ be a partially ordered set such that all totally ordered subsets (called *chains*) have an upper bound, then $(X, \leqslant)$ has a maximal element. The goal of the exercise is to prove the following statement:

Every set can be well-ordered.

Let $(X, \leqslant_X)$ be an ordered set. We define the following set

$$\mathscr{A} = \{(A, \leqslant) \mid A \subset X \text{ and } \leqslant \text{ is a well-ordering of } A\}\,.$$

1. We define the following relation on $\mathscr{A}$,

$$(A_1, \leqslant_1) \leqslant_{\mathscr{A}} (A_2, \leqslant_2) \iff A_1 \text{ is an initial segment of } A_2 \text{ and for all } x,\, y \in A_1 \subset A_2,\, x \leqslant_1 y \Leftrightarrow x \leqslant_2 y.$$

Prove that $(\mathscr{A}, \leqslant_{\mathscr{A}})$ is an ordered set. What is the minimal element of $(\mathscr{A}, \leqslant_{\mathscr{A}})$.

2. Show that the maximal elements of $(\mathscr{A}, \leqslant_{\mathscr{A}})$ are exactly the well-orderings $(X, \leqslant)$ of X.

3. Our goal is now to show that every chain has an upper bound. Let $\mathscr{C}$ be a chain of $\mathscr{A}$ and consider

$$Y = \bigcup_{(A, \leqslant) \in \mathscr{C}} A.$$

a. Is Y a subset of $\mathscr{A}$, X or $\mathscr{C}$?

b. We define a relation on Y:

$$y \leqslant_Y y' \iff \text{There exists } (A, \leqslant) \in \mathscr{C} \text{ such that } y,\, y' \in A \text{ and } y \leqslant y'.$$

Show that $(Y, \leqslant_Y)$ is a well-ordered set.

c. Prove that $(Y, \leqslant_Y)$ is an upper bound for $\mathscr{C}$.

4. Prove the well-ordering theorem using Zorn's lemma. Note that you need to prove that every set can be partially ordered.

Exercise 6. *On the axiom of choice.* Prove that the well-ordering theorem implies the axiom of choice: For every set X we can construct a function $f : \mathcal{P}(X) \to X$ such that for $A \subset X$, $f(A) \in A$

Note that with the previous homework you just proved that the axiom of choice, Zorn's lemma and the well-ordering theorem are equivalent:

$$\text{Axiom of choice} \Rightarrow \text{Zorn's lemma} \Rightarrow \text{Well-ordering theorem} \Rightarrow \text{Axiom of choice.}$$

Thus, you would just need to prove one of these results to get the others. However, it has been showed that they cannot be proved using the *basic tools* of mathematics (called the Zermelo–Fraenkel axioms) by Kurt Gödel and Paul Cohen. Mathematicians then have to admit one of these results to use them in other proofs.

Another remark comes from the possible construction of well-orderings. By the well-ordering theorem, we know that a well-ordering of the real numbers exists. However, we do not know how to actually construct it or give it a proper definition. A problem if you are a *constructivist*...

Kurt Gödel
(1906–1978)

Paul Cohen
(1934–2007)