Homework 4
Math 16110-Section 50

Due: Tuesday October 29th

The goal of this homework is to prove Zorn's lemma (or Kuratowski–Zorn's lemma) which actually has a lot of applications you will most likely see in further mathematical adventures. We first give a definition of an order, a total order and a well-ordered set. The definition of ordering we give here is slightly different than in the script as we consider reflexive ordering: we always have $x \leqslant x$.

Definition 1. An order is a binary relation $\leqslant$ on some set X which follows the three following properties:

(i) (Reflexivity) For every $x \in X$, $x \leqslant x$.

(ii) (Antisymmetry) For every x, $y \in X$, if $x \leqslant y$ and $y \leqslant x$ then $x = y$.

(iii) (Transitivity) For every x, y, $z \in X$, if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

$(X, \leqslant)$ is called an *ordered set*. Besides, if for all x, $y \in X$, we have either $x \leqslant y$ or $y \leqslant x$ then we call $(X, \leqslant)$ a *totally ordered set*. In other words, any couple of elements of X are comparable.

1. Consider $\mathcal{P}(\{0,1\}) = \{\emptyset, \{0,1\}, \{0\}, \{1\}\}$. Show that $(\mathcal{P}(\{0,1\}), \subset)$ is an ordered set but not a totally ordered set.

Definition 2. $(X, \leqslant)$ is a *well-ordered set* if $(X, \leqslant)$ is a totally ordered set and if every non-empty subset of X has a least element with respect to $\leqslant$.

For instance we saw in Script 0 that $\mathbb{N}$ with the usual ordering is well-ordered (via the well-ordering principle).

2. We define the following relation $\leqslant_{\mathbb{Z}}$ on the integers, where $\leqslant$ is the usual ordering,

$$x \leqslant_{\mathbb{Z}} y \iff (|x| < |y| \text{ or } (|x| = |y| \text{ and } x \leqslant y)).$$

a. Prove that $(\mathbb{Z}, \leqslant_{\mathbb{Z}})$ is a well-ordered set.

b. Give the least element of $\{2n \mid n \in \mathbb{Z} \setminus \{0\}\}$.

As we saw in Definition 1, a set could be an *ordered set* but not a *totally ordered set*. However, a set always has subsets that are totally-ordered and we call them *chains*.

Definition 3. Let $(X, \leqslant)$ be an ordered set. A *chain* is a totally ordered subset of $(E, \leqslant)$.

3. Let $(X, \leqslant)$ be an ordered set. Prove that $\emptyset$ and $\{x\}$ for $x \in X$ are chains of X.

We are now ready to state Zorn's lemma. The goal of the exercise is now to prove it.

Lemma 4 (Zorn's lemma)**.** *Let $(X, \leqslant)$ be an ordered set whose every well-ordered chain has an upper bound then $(X, \leqslant)$ has a maximal element: there exists $m \in X$ such that for all $x \in X$ $(m \leqslant x \Rightarrow m = x)$.*

Before starting the proof of the lemma, we give another definition.

Definition 5. Let $(X, \leqslant)$ be an ordered set. An *initial segment* of $(X, \leqslant)$ is a subset I of X such that if $x \in I$ then $\{y \in X \mid y < x\} \subset I$.

For instance $(-\infty, 2)$ is an initial segment of $\mathbb{R}$ with the usual ordering and $\{1, 2, 3\}$ is an initial segment of $\mathbb{N}$ with the usual ordering.

4. Let $(X, \leqslant)$ be an ordered set and I_1 and I_2 be two initial segments of $(X, \leqslant)$. Show that $I_1 \cap I_2$ and $I_1 \cup I_2$ are also initial segments of $(X, \leqslant)$.

5. Let $(C, \leqslant)$ be a well-ordered set. Prove that for every initial segment I of $(C, \leqslant)$ such that $I \neq C$, there exists a unique $x \in C$ such that $I = \{y \in C \mid y < x\}$.

6. Let $(X, \leqslant)$ be an ordered set and $\mathscr{C}$ a set of well-ordered chains of $(X, \leqslant)$ with the following property

$$\text{For all chains } C, D \in \mathscr{C}, \text{ either } C \text{ is an initial segment of } D \text{ or } D \text{ is an initial segment of } C. \tag{0.1}$$

a. Show that every element of $\mathscr{C}$ is an initial segment of $\bigcup_{C \in \mathscr{C}} C$.

b. Show that $\bigcup_{C \in \mathscr{C}} C$ is a well-ordered chain of $(X, \leqslant)$.

For a set X we consider a function $f : \mathcal{P}(X) \to X$ such that for every $A \in \mathcal{P}(X)$ we have $f(A) \in A$. In other words, this is a function f which "chooses" an element out of every subset of X[1].

7. From the function f, construct a function g defined on every well-ordered chain of $(X, \leqslant)$ which has a strict upper bound, and such that $g(C)$ is a strict upper bound of C (a *strict upper bound* of C is an upper bound m of C such that $m \notin C$).

8. Given this function g, we define a *g-chain* as a well-ordered chain C of $(X, \leqslant)$ such that:

$$\text{For all } x \in C,\ x = g\left(\{y \in C \mid y < x\}\right).$$

a. Prove that if C is a g-chain which has a strict upper bound then $C \cup \{g(C)\}$ is also a g-chain.

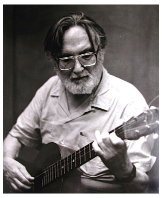
b. Show that if C and D are two g-chains then one is an initial segment of the other.
Hint: One could consider $\mathscr{C}$ to be the set of all the well-ordered chains that are initial segments of both C and D and study $\bigcup_{E \in \mathscr{C}} E$

9. We are now ready to prove Zorn's lemma. We suppose the hypothesis of the lemma true in this question. We will combine most of the previous questions to obtain the result.

a. Using **Question 8.**, show that the sets of g-chains follows the property (0.1).

b. Using **Question 6.**, show that the reunion of all the g-chains of $(X, \leqslant)$ is also a g-chain.

c. Prove Zorn's lemma.



Max August Zorn
(1906–1993)

Kazimierz Kuratowski
(1896–1980)

[1]The existence of such a function actually cannot be proved in all generality, this is called the *axiom of choice*