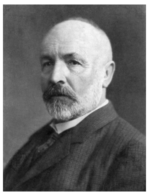

Georg Cantor

Homework 2
Math 16110-Section 50

Due: Tuesday October 15th

Felix Bernstein

Exercise 1. *On Cantor's theorem.* Let A be any set. We recall the definition of the powerset

$$\mathcal{P}(A) = \{B,\, B \subset A\}\,.$$

1. Show that there is no bijection between A and $\mathcal{P}(A)$.
Hint: For a function $f : A \to \mathcal{P}(A)$, consider the set $\{a \in A, a \notin f(a)\}$.

2. Show that there is no injective function from $\mathcal{P}(\mathbb{N})$ to $\mathbb{N}$.

Exercise 2. *Beware of countability.* Let $n \in \mathbb{N}$

1. Show that $\{0,1\}^2$ is finite countable, that $\{0,1\}^n$ is finite countable but that $\{0,1\}^{\mathbb{N}}$ is uncountable.

Exercise 3. *Constructing some bijections.*

1. Construct a bijection from $\mathbb{N}$ to $\mathbb{Z}$.

2. Construct a bijection from $\{1/n,\, n \geqslant 1\}$ to $\{1/n,\, n \geqslant 2\}$.

3. By using the previous question, construct a bijection from $[0,1]$ to $[0,1)$.
Hint: You can write $[0,1] = \{1/n,\, n \geqslant 1\} \cup A$ for some set A.

Exercise 4. *Function on the powerset.* Let E be a set, we denote $\mathcal{P}(E)$ the set of his subsets as in Exercise 1. Consider A and B two subsets of E. We define

$$\begin{array}{rccc} f: & \mathcal{P}(E) & \to & \mathcal{P}(A) \times \mathcal{P}(B) \\ & X & \mapsto & (X \cap A,\, X \cap B). \end{array}$$

1. Show that f is an injection if and only if $A \cup B = E$.

2. Show that f is a surjection if and only if $A \cap B = \emptyset$.

3. Find a condition both necessary and sufficient on A and B for f to be a bijection. In this case, give the function inverse of f.

Exercise 5. *On the Cantor-Bernstein theorem.* Let E and F be two sets. The goal of this exercise is to show that if there exists an injection $f : E \to F$ and an injection $g : F \to E$ than there exists a bijection from E to F. We define the following family of sets

$$A_0 = E \setminus g(F),\ A_1 = (g \circ f)(A_0)\,, \ldots,\ A_{n+1} = (g \circ f)(A_n),$$
$$B = \bigcup_{n \geqslant 0} A_n, \quad \text{and} \quad C = E \setminus B.$$

1. Constructing the application

- **a.** Show that for all $x \in C$ there exists a unique $z \in F$ such that $x = g(z)$. We will denote this element $\phi(x)$.
- **b.** For $x \in B$, we denote $\phi(x) = f(x)$. Show that $\phi : E \to F$ is well defined.

2. ϕ is an injection.

- **a.** Show that the restrictions of ϕ to B and to C are injections.
- **b.** Consider $x \in C$ and $y \in B$ such that $\phi(x) = \phi(y)$. Show that $x = (g \circ f)(y)$.
- **c.** Deduce that ϕ is an injection.
  Hint: Show that by the definition of A_n's, $\phi(x) = \phi(y)$ is impossible for $x \in C$ and $y \in B$.

3. Show that ϕ is surjective.
Hint: For $z \in F$, split the problem in two cases, if $g(z) \in C$ or if $g(z) \in B$.

4. Example: For $E = \mathbb{N}$, $F = \{2,3,\ldots\}$, $f : E \to F, n \mapsto n+4$, $g : F \to E$, $n \mapsto n$, give the sets A_n, B, C, and the application ϕ.