Last Name: ________________________________

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MATH 20400 Section 51
Analysis in $\mathbb{R}^n$

Final Exam

March 16th 2020

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Directions: You have 2 hours for this exam. No calculators, notes, books, laptops, phones, etc. are allowed. If you see an error in the exam, correct it in your solution and explain why you made the correction. No questions will be answered during the exam. Every answer has to be justified and proofs must be complete.

Initials:____

Exercise 1. Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^3}{\sqrt{x^4+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that f is continuous on $\mathbb{R}^2$.

2. Compute the partial derivatives of f at $(0,0)$.

3. Show that f is not differentiable at $(0,0)$.

Initials:____

Exercise 2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

1. Find the largest subset $U \subset \mathbb{R}^3$ onto which f is C^2.

2. Show that for $(x, y, z) \in U$ we have

$$\partial_1^2 f(x, y, z) + \partial_2^2 f(x, y, z) + \partial_3^2 f(x, y, z) = 0.$$

Initials:____

Exercise 3. Let $c \in \mathbb{R}$. Find the local and global extrema of

1. $f_c : \mathbb{R}^2 \to \mathbb{R}$ defined by $f_c(x, y) = x^3 + y^3 - 3cxy$
2. $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = x^2y^3$.

Initials:____

Exercise 4. We define the following hypersurface

$$\mathscr{S} = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 - \frac{2}{3}zx - 2zy + 3z = \frac{9}{2} \right\}$$

1. Prove that $\mathscr{S}$ is a smooth hypersurface.

2. Describe geometrically the set of points $M \in \mathscr{S}$ such that the tangent plane to $\mathscr{S}$ at M contains the point $(0, 0, 0)$.

Initials:____

Exercise 5. Find the global extrema of $f(x, y) = 2x - y$ under the constraint $x^2 + xy - y^2 = 1$.

Initials:____

Exercise 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2y + 3y - 2$

1. Give the partial derivatives of f and use Taylor's formula to develop $f(x, y)$ in terms of power of $(x - 1)$ and $(y + 2)$.
2. Develop $f(x, y) = f(x - 1 + 1, y + 2 - 2)$ to find the same result.

Initials:____

Exercise 7. We consider the following system of equations for $(x, y, z, t) \in \mathbb{R}^4$,

$$\begin{cases} 2x + 3y + 5x^2y^3 - \sin(z) + z^3y = 0, \\ x - y + \sin(x^6y^3) - \tan(t) = 0. \end{cases}$$

1. Find one explicit solution (x_0, y_0, z_0, t_0) to the system.

2. Show that around the point (x_0, y_0, z_0, t_0) we can express (z_0, t_0) as a C^1 function φ of (x_0, y_0).

3. Compute the derivative $\partial_j \varphi_i(z_0, y_0)$ for $i, j \in \{1, 2\}$.

Initials:____

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$ be C^1. Suppose there exists $k \in (0, 1)$ such that for all $x \in \mathbb{R}$, $|f'(x)| \leqslant k$. We define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = (x + f(y), y + f(x)).$$

1. Show that g is a diffeomorphism from $\mathbb{R}^2$ to $g(\mathbb{R}^2)$.

2. Show that $g(\mathbb{R}^2) = \mathbb{R}^2$.

Initials:____

Exercise 9. **1.** Show by induction that the following integrals exist and compute them:

$$I_n = \int_0^{+\infty} t^n \mathrm{e}^{-t} \mathrm{d}t.$$

2. Show that the following integral exists and compute it

$$I = \int_0^{+\infty} \frac{\arctan t}{1+t^2} \mathrm{d}t.$$