Homework 9
Math 20400-Section 51

Due: Monday March 9th

Exercise 1. Find all continuous functions $f : [0,1] \to [0,1]$ such that $\int_0^1 f(t)\mathrm{d}t = \int_0^1 f(t)^2\mathrm{d}t$.

Exercise 2. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$. Suppose that there exists $x \in [a,b]$ such that $f(x) \neq 0$ and that there exists $n \in \mathbb{N}$ such that for all $k \leqslant n$, $\int_a^b t^k f(t)\mathrm{d}t = 0$. We want to prove that there exists $n+1$ distinct points in $[a,b]$ where f vanishes and changes sign.

1. Study the case $n = 0$ and $n = 1$.

2. Prove the statement for all n.

Exercise 3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For all $x \in \mathbb{R}$ we define

$$g(x) = \int_0^1 f(t)\mathrm{e}^{tx}\mathrm{d}t.$$

Prove that g is continuous on $\mathbb{R}$.

Exercise 4. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$2yf(x) = \int_{x-y}^{x+y} f(t)\mathrm{d}t.$$

Exercise 5. Find the limits of the following sequences

1. $u_n = n\left(\dfrac{1}{(n+1)^2} + \cdots + \dfrac{1}{(n+n)^2}\right)$.

2. $u_n = \sqrt[n]{\left(1+\left(\dfrac{1}{n}\right)^2\right)\left(1+\left(\dfrac{2}{n}\right)^2\right)\cdots\left(1+\left(\dfrac{n}{n}\right)^2\right)}$.

3. $u_n = \dfrac{1}{n}\prod_{k=1}^{n}(k+n)^{1/n}$.

4. $u_n = \sum_{p=n}^{2n} \dfrac{1}{p}$.

Exercise 6. Let $f : [a,b] \to \mathbb{R}$ be continuous and $g : \mathbb{R} \to \mathbb{R}$ be continuous and convex. Show that

$$g\left(\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right) \leqslant \frac{1}{b-a}\int_a^b g(f(t))\mathrm{d}t.$$

Exercise 7. Let $f : [a,b] \to \mathbb{R}$ be $C^2([a,b])$. We define $I := \int_a^b f(t)\mathrm{d}t$ and $I_m := (b-a)f(\frac{a+b}{2})$. Denote $M_2 := \max\{|f''(x)| \mid x \in [a,b]\}$.

1. Let $\Delta(x) := \int_{c-x}^{c+x} f(t)\mathrm{d}t - 2xf(c)$ where $c = \frac{a+b}{2}$. Show that for all $x \in [0, \frac{b-a}{2}]$, $|\Delta''(x)| \leqslant 2xM_2$.

2. Prove that $|I - I_m| \leqslant M_2\frac{(b-a)^3}{24}$.

3. For all $n \geqslant 1$, denote $I_{m,n} = \frac{b-a}{n}\sum_{k=0}^{n-1} f(\frac{x_k + x_{k+1}}{2})$ with $x_k = a + k\frac{b-a}{n}$. Show that

$$\left|\int_a^b f(t)\mathrm{d}t - I_{m,n}\right| \leqslant \frac{(b-a)^3}{24n^2}M_2.$$

Georg Friedrich Bernhard Riemann
(1826–1866)

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(1859–1925)