Homework 8
Math 20400-Section 51

Due: Monday March 2nd

Exercise 1. Prove that the map $\phi : (U, v) \to (u+v, uv)$ is a C^1-diffeomorphism from $U = \{(u, v) \in \mathbb{R}^2 \mid u > v\}$ to some open set you should determine.

Exercise 2. Let $a, b \in \mathbb{R}$ such that $|ab| < 1$.

1. Let $v \in \mathbb{R}$. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x + a\sin(v - b\sin(x))$. Show that g is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

2. Denote $f(x, y) = (x + a\sin(y), y + b\sin(x))$. Show that f is a C^1-diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Exercise 3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 such that there exists $k > 0$ such that for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\|f(x) - f(y)\| \geqslant k\|x - y\|.$$

1. Prove that $f(\mathbb{R}^n)$ is a closed set of $\mathbb{R}^n$.

2. Prove that df_x is invertible at every point of $x \in \mathbb{R}^n$.

3. Prove that f is a C^1-diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Exercise 4. Define the following map

$$\Phi : \begin{cases} \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ (r, \theta) & \longmapsto & (r\cos\theta, r\sin\theta). \end{cases}$$

1. Is Φ a diffeomorphism?

2. Compute the derivative $d\Phi_{(r,\theta)}$ for all r and θ and show that it is invertible if $r \neq 0$.

3. Denote $\Omega = \mathbb{R}_+ \setminus \{0\} \times (-\pi, \pi) \subset \mathbb{R}^2$. Show that $\Phi_{|\Omega} : \Omega \to \Phi(\Omega)$ is a C^∞-diffeomorphism.

Exercise 5. **1.** Show that the relation $x + y + z + \sin(xyz) = 0$ defines z as a function of x and y around the point $(0, 0, 0)$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0)$.

2. Show that the relations $x + y - zt = 0$, $xy - z + t = 0$ define two implicit functions $x = \varphi_1(z, t)$ and $y = \varphi_2(z, t)$ around $(0, 1)$ with $\varphi_1(0, 1) = 1$. Compute the derivative of φ_1 and φ_2 at this point.

Exercise 6. Let P_0 be a real polynomial of degree d and α_0 be a simple root of P_0.

1. By considering the map $f : \mathbb{R}_d[X] \times \mathbb{R} \to \mathbb{R}$ defined by $f(P, \alpha) = P(\alpha)$, show that there exists a neighborhood $U \times V$ of (P_0, α_0) such that every polynomial $P \in U$ has a unique root $\alpha \in V$, that this root is simple and that the map $P \mapsto \alpha$ is C^1.

2. Show that if P_0 has k simple roots, there exists a neighborhood U of P_0 and k C^1 functions $\phi_1, \ldots, \phi_k : U \to \mathbb{R}$ such that for all $P \in U$, $\phi_1(P), \ldots, \phi_k(P)$ are distinct and are simple roots of P.

3. What happens if α_0 is not a simple root?

Augustin-Louis Cauchy
(1789–1857)

Ulisse Dini
(1845–1918)