Homework 7
Math 20400-Section 51

Due: Monday February 24th

Exercise 1. Consider a convex polygon with n sides inscribed in a circle. We denote P its perimeter and $\mathrm{e}^{\mathrm{i}a_1}, \dots, \mathrm{e}^{\mathrm{i}a_n}$ its vertices with $0 \leqslant a_1 < a_2 < \cdots < a_n < 2\pi$.

1. Show that

$$P = 2\sum_{k=1}^{n-1} \sin\left(\frac{a_{k+1} - a_k}{2}\right) + \sin\left(\frac{a_1 + 2\pi - a_n}{2}\right)$$

2. Show that P is maximal when the polygon is regular.

Exercise 2. We want to build a box which has a rectangular cuboid shape without a lid on top. The total volume of the box must be $0.5m^3$ and to optimize the amount of material needed, we want that the sum of the areas of the faces must be as small as possible. What dimensions should we choose to make this box?

Exercise 3. Study the extrema of the function $f_a : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \mathrm{e}^{axy}$ with $a > 0$ under the constraint that $x^3 + y^3 + x + y - 4 = 0$.

Exercise 4. Let $n \geqslant 2$ and $f : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) = \prod_{i=1}^n x_i$. Denote $\Gamma = \left\{(x_1, \dots, x_n) \in \mathbb{R}_+^n \mid x_1 + \cdots + x_n = 1\right\}$

1. Show that f admits a global maximum on Γ and find it.

2. Prove the inequality of arithmetic and geometric means: for all $(x_1, \dots, x_n) \in \mathbb{R}_+^n$ we have

$$\prod_{i=1}^n x_i^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^n x_i.$$

Exercise 5. In this exercise, we study the determinant function $\det : \mathbb{R}^{n^2} \to \mathbb{R}$, defined by

$$\det((M_{ij})_{1\leqslant i,j\leqslant n}) = \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) M_{1\sigma(1)} \dots M_{n\sigma(n)}.$$

1. We first want to compute the derivative of det.

a. Show that det is $\mathcal{C}^\infty(\mathbb{R}^{n^2})$ and that the (i, j)-th partial derivative of det is

$$\partial_{M_{ij}} \det(M) = \sum_{\sigma\in\mathfrak{S}_n(i,j)} \epsilon(\sigma) M_{1\sigma(1)} \dots M_{(i-1)\sigma(i-1)} M_{(i+1)\sigma(i+1)} \dots M_{n\sigma(n)}$$

where $\mathfrak{S}_n(i, j)$ is the set of bijections from $\{1, \dots, n\} \setminus \{i\}$ to $\{1, \dots, n\} \setminus \{j\}$.

b. Prove that, if we denote $M_j = (M_{ij})_{i=1\dots n}$ the j-th column of the matrix M,

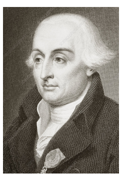
$$\mathrm{d}(\det)_{(M_{ij})}(H_{ij}) = \sum_{i=1}^n \det(M_1, \dots, H_i, \dots M_n).$$

2. We denote $X = \{(V_1, \dots, V_n) \in \mathbb{R}^{n^2} \mid \|V_1\| = \cdots = \|V_n\| = 1\}$. Prove that $\max_{V\in X} \det V \geqslant 1$.

3. Prove that the maximum of det on X can only be attained on an orthonormal basis of $\mathbb{R}^n$ and that $\max_{V\in X} \det V = 1$.

4. Prove that

$$|\det(V_1, \dots, V_n)| \leqslant \|V_1\| \dots \|V_n\|.$$



Joseph-Louis Lagrange
(1736–1813)

Jacques Salomon Hadamard
(1865–1963)