Homework 6
Math 20400-Section 51

Due: Monday February 17th

Exercise 1. Find the local and global extrema of the following functions

1. $f(x,y) = 2x^3 + 6xy - 3y^2 + 2$. **2.** $f(x,y) = y(x^2 + (\log y)^2)$
3. $f(x,y) = x^4 + y^4 - 4xy$. **4.** $f(x,y) = x^2 + y^3$
5. $f(x,y) = x^4 + y^3 - 3y - 2$.

Exercise 2. We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if for all $x, y \in \mathbb{R}^n$ we have

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) \quad \forall t \in [0,1].$$

Prove that if f is a convex function differentiable on $\mathbb{R}^n$ then any critical point of f is a global minimum.

Exercise 3. Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is *coercive* if for all $A \in \mathbb{R}_+$ there exists a $B \in \mathbb{R}_+$ such that if $\|x\| \geqslant B$ then $f(x) \geqslant A$. Show that if f is a continuous and coercive function from $\mathbb{R}^n$ to $\mathbb{R}$ then f admits a global minimum.

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 in a neighborhood of 0 and such that $f(0) = 0$ and $f'(0) \neq 0$. We define the function $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x,y) = f(x)f(y)$.

1. Show that F does not admit a local extremum at $(0,0)$, what type of point is $(0,0)$ for the function F?

2. Find the local extrema of the function

$$f(x,y) = \sin(2\pi x)\sin(2\pi y).$$

Exercise 5. Let $(x_i, y_i)_{i=1}^n$ be a family of points in $\mathbb{R}^2$. The *least squares regression line* is the line of equation $y = mx + p$ which minimizes the quantity

$$F(m,p) = \sum_{k=1}^{n} (y_k - mx_k - p)^2.$$

In this exercise, we denote $\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$ and $\bar{y} = \frac{1}{n}\sum_{k=1}^{n} y_k$.

1. Show that if (m,p) is a point where F admits this minimum then (m,p) is a solution to the equations

$$\sum_{k=1}^{n} (y_k - mx_k - p) = 0 \quad \text{and} \quad \sum_{k=1}^{n} x_k(y_k - mx_k - p) = 0.$$

2. By solving this system, prove that if (m,p) is a point where F admits its minimum and $\sum_{k=1}^{n}(x_k - \bar{x})^2 \neq 0$ then (m,p) is unique and

$$m = \frac{\sum_{k=1}^{n}(x_k - \bar{x})(y_k - \bar{y})}{\sum_{k=1}^{n}(x_k - \bar{x})^2} \quad \text{and} \quad p = \bar{y} - m\bar{x}.$$

One can also show that this minimum actually exists by using **Exercise 4.** for instance but this is not asked in this assignment.

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(1811–1874)

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