Homework 4
Math 20400-Section 51

Due: Monday February 3rd

Exercise 1. *Counter-example to Schwarz's theorem.* We define the function $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{otherwise.}\end{cases}$$

We saw in the last homework that the function f was $C^1(\mathbb{R})$. Prove that f is not C^2.

Exercise 2. *The converse of Schwarz's theorem is not true.* We define the function $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,y)=\begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{otherwise.}\end{cases}$$

1. Prove that $\partial_{12}f(0,0)=\partial_{21}f(0,0)=0$.

2. Show that the function f does not follow the assumptions from Schwarz's theorem at $(0,0)$.

Exercise 3. Compute the Taylor polynomial of degree 3 for the following functions at the specified point in an arbitrary direction $h\in\mathbb{S}^2$,

1. $f(x,y,z)=\dfrac{1}{xyz}$ at $(1,1,1)$

2. $f(x,y,z)=\mathrm{e}^{xy+yz}$ at $(0,0,0)$.

Exercise 4. *A follow-up on homogeneous functions.* We suppose that f is a homogeneous function of degree r as in the last homework (and we can use the results from the last homework):

$$\forall (x,y)\in\mathbb{R}^2,\quad \forall t>0,\quad f(tx,ty)=t^rf(x,y).$$

Prove that if f is $C^2(\mathbb{R}^2)$ then for all $(x,y)\in\mathbb{R}^2$,

$$x^2\partial_1^2f(x,y)+2xy\partial_{12}f(x,y)+y^2\partial_2^2f(x,y)=r(r-1)f(x,y).$$

Exercise 5. *On harmonic functions.* A function $f:\mathbb{R}^2\to\mathbb{R}$ is called *harmonic* if we have for all $(x,y)\in\mathbb{R}^2$,

$$\partial_1^2f(x,y)+\partial_2^2f(x,y)=0.$$

Let f be an harmonic function.

1. We suppose that f is $C^3(\mathbb{R}^2)$. Show that $\partial_1 f$, $\partial_2 f$ and $(x,y)\mapsto x\partial_1f(x,y)+y\partial_2f(x,y)$ are harmonic functions.

2. We suppose that f is a radial function: there exists $\varphi:\mathbb{R}\to\mathbb{R}$ a C^2 function such that $f(x,y)=\varphi(x^2+y^2)$ for all $(x,y)\in\mathbb{R}^2$. Show that for $t>0$ we have

$$\varphi'(t)+t\varphi''(t)=0.$$

3. We admit that the only solutions to the differential equation $xy'(x)+y(x)=0$ are functions of the form $y(x)=\frac{A}{x}$. Find all radial harmonic functions.

Hermann Amandus Schwarz
(1843–1921)

Alexis Claude Clairaut
(1713–1765)