Homework 3
Math 20400-Section 51

Due: Monday January 27th

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{C}^1$. Compute the derivative of the following functions

1. $g(x, y) = f(y, x)$ **2.** $g(x) = f(x, x)$
3. $g(x, y) = f(y, f(x, x))$ **4.** $g(x) = f(x, f(x, x))$.

Exercise 2. Show that the following function admits a directional derivative in any direction at $(0, 0)$ but is not continuous at $(0, 0)$.

$$g(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 3. After justifying that the following functions are differentiable, compute their Jacobian matrix

1. $f : \mathbb{R}^3 \to \mathbb{R}^2$ with $f(x, y, z) = \left(\frac{1}{2}(x^2 - z^2), \sin(x)\sin(y)\right)$.

2. $f : \mathbb{R}^2 \to \mathbb{R}^3$ with $f(x, y) = \left(xy, \frac{1}{2}x^2 + y, \log(1 + x^2).\right)$

Exercise 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

1. Is f continuous on $\mathbb{R}^2$?

2. Is f $\mathcal{C}^1$ on $\mathbb{R}^2$?

3. Is f differentiable on $\mathbb{R}^2$?

Exercise 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an application $\mathcal{C}^1$ on $\mathbb{R}^2$ and $r \in \mathbb{R}$. We say that f is homogeneous of degree r if

$$\forall (x, y) \in \mathbb{R}^2,\ \forall t > 0,\ f(tx, ty) = t^r f(x, y).$$

1. Give an example of an homogeneous function of degree 3.

2. Show that if f is homogeneous of degree r then its partial derivatives are homogeneous of degree $r - 1$.

3. Show that if f is homogeneous of degree r then

$$\forall (x, y) \in \mathbb{R}^2,\ x\partial_1 f(x, y) + y\partial_2 f(x, y) = rf(x, y).$$

4. We want to show the other implication, we suppose that f follows the previous identity (called the Euler identity) and we want to show that f is homogeneous of degree r. Consider $\varphi(t) = f(tx, ty)$, by differentiating the function $t \mapsto t^{-r}\varphi(t)$, show that f is homogeneous of degree r.

Carl Gustav Jacob Jacobi
(1804–1851)

Leonhard Euler
(1707–1783)