Homework 2
Math 20400-Section 51

Due: Wednesday January 22nd

Exercise 1. *On the Landau–Kolmogorov inequality.* Let $f \in C^2(\mathbb{R})$. We suppose that f and f'' are bounded and we denote

$$M_0 := \sup_{x\in\mathbb{R}} |f(x)| \quad \text{and} \quad M_2 := \sup_{x\in\mathbb{R}} |f''(x)|.$$

The goal of this exercise is to prove that f' is bounded and to bound $M_1 = \sup_{x\in\mathbb{R}} |f'(x)|$ in terms of M_0 and M_2.

1. Prove that we have for all $x \in \mathbb{R}$ and $h \in \mathbb{R}$,

$$|f(x+h) - f(x) - hf'(x)| \leqslant \frac{M_2}{2}h^2.$$

2. Prove that we have for all $x \in \mathbb{R}$,

$$|f'(x)| \leqslant \frac{2}{h}M_0 + \frac{h}{2}M_2$$

Note that if we take $h = 1$ for instance, we proved that $M_1 \leqslant 2M_0 + \frac{M_2}{2}$. We can however obtain a better bound.

3. By studying the function $h \mapsto \frac{2}{h}M_0 + \frac{h}{2}M_2$, prove that

$$M_1 \leqslant 2\sqrt{M_0 M_2}.$$

Exercise 2. Do the following functions have a limit at 0?

1. $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ **2.** $f(x,y,z) = \frac{xy+yz}{x^2+2y^2+3z^2}$

3. $f(x,y) = \left(\frac{x^2+y^2-1}{x}\sin x, \frac{\sin(x^2)+\sin(y^2)}{\sqrt{x^2+y^2}}\right)$ **4.** $f(x,y) = \frac{1-\cos(xy)}{xy^2}$.

Exercise 3. Let $\alpha, \beta > 0$. Say, according to the values of α and β, if the function

$$f(x,y) = \frac{x^\alpha y^\beta}{x^2+y^2}$$

has a limit at $(0,0)$ and prove it.

Exercise 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that for all $(x,y) \in (\mathbb{R}^2)^2$ we have

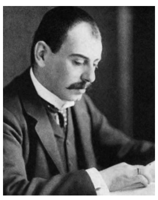
$$|f(x) - f(y)| \leqslant \|x-y\|^2.$$

Prove that f is a constant function.

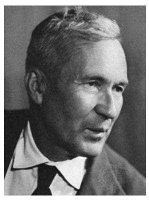
Exercise 5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. We suppose that for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, $f(\lambda x) = \lambda f(x)$.

1. Show that $f(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$.

2. Show that f is a linear map.



Edmund Landau
(1877–1938)



Andrey Kolmogorov
(1903–1987)