Homework 1
Math 20400-Section 51

Due: Monday January 13th

Exercise 1. Calculate the derivatives of these functions using the definition

1. $f : x \mapsto x^n$ with $n \in \mathbb{N}$

2. $f : x \mapsto \sin(x)$.

3. $f : x \mapsto \tan(x)$. We recall the following:

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \quad \text{and} \quad \lim_{h\to 0} \frac{\tan(h)}{h} = 1.$$

Exercise 2. On which subsets of $\mathbb{R}$ these two functions are continous? differentiable?

$$f : x \mapsto \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases} \qquad g : x \mapsto \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 3. Prove that we have for all $x \in \mathbb{R}$,

$$\arctan(\mathrm{e}^x) = \arctan\left(\mathrm{th}\left(\frac{x}{2}\right)\right) + \frac{\pi}{4}, \quad \text{we recall} \quad \mathrm{th}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Exercise 4. Find all differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall (x,y) \in \mathbb{R}^2, \quad f(x+y) = f(x) + f(y).$$

Exercise 5. Let f be a function defined on an interval I and let $a \in \mathrm{Int}(I)$. We define, if they exist, these limits:

$$f'_r(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \quad f'_\ell(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad f'_s(a) = \lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}.$$

1. Show that if $f'_r(a)$ and $f'_\ell(a)$ exist then $f'_s(a)$ exists and compute it.

2. If $f'_s(a)$ exists, does it mean that $f'_r(a)$ and $f'_\ell(a)$ exist?

Exercise 6. Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be a function in $C^n(I)$ such that f vanishes at $n+1$ distinct points in I.

1. Show that the n-th derivative of f vanishes at, at least, one point in I.

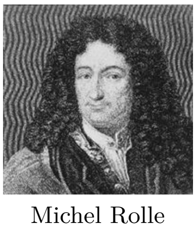
2. Let $\alpha \in \mathbb{R}$. Show that the $(n-1)$-th derivative of $f' + \alpha f$ vanishes at, at least, one point in I.
Hint: One can introduce an auxiliary function.

Exercise 7. Show that

$$\forall x > 0, \quad \frac{1}{1+x} < \log(1+x) - \log(x) < \frac{1}{x}.$$

Find, for all $k \in \mathbb{N} \setminus \{0,1\}$,

$$\lim_{n\to\infty} \sum_{p=n+1}^{kn} \frac{1}{p}.$$



Michel Rolle
(1652–1719)

Bernard Bolzano
(1781–1848)