HOMEWORK 8
MATH 18500-SECTION 41, 51
DUE: MAY 27TH

Exercise 1. Find the following Laplace transforms and inverse Laplace transforms. Do not give answers involving complex numbers.

a. $\mathcal{L}\{f(t)\}$, where $f(t) = e^{-t} + \cos(2t) + 3t^4e^{5t} + 6te^{-7t}\sin(8t) - 9\delta(10-t)$.

b. $\mathcal{L}\{g(t)\}$, where $g(t)$ is defined by

$$g(t) = \begin{cases} 1 & t \leq 1 \\ 2t-1 & 1 \leq t \leq 2 \\ 5-t & 2 \leq t \leq 3 \\ 2 & t \geq 3 \end{cases}$$

Also sketch the graph of $g(t)$ - you will see that it is a continuous function.

c. $\mathcal{L}^{-1}\{F(s)\}$, where $F(s) = \dfrac{1+s^2}{8-s^3}$.

d. $\mathcal{L}^{-1}\{G(s)\}$, where $G(s) = \dfrac{1}{(1-s^2)^2}$.

Exercise 2. Solve the following system of first order equations:

$$\begin{cases} \dfrac{dx}{dt} &= 2x + y + e^{-t}\ , \ \ x(0) = 1 \\[2ex] \dfrac{dy}{dt} &= 2x + 3y - e^{t}\ , \ \ y(0) = 2 \end{cases}$$

Note: This system cannot be solved using the standard methods from week 3, because it is not autonomous. To solve it, take the Laplace transforms of both equations and solve the resulting system of equations for $X(s)$ and $Y(s)$. Then apply the inverse Laplace transform to obtain $x(t)$. To obtain $y(t)$, you can either take the inverse Laplace transform of $Y(s)$, or solve the first equation for $y(t)$ in terms of $x(t)$.

Exercise 3. Consider a damped oscillator which is modelled by the equation

$$y'' + 3y' + 2y = f(t)$$

a. Find the transfer function of the oscillator, i.e. the ratio of Laplace transforms $H(s) = \frac{Y(s)}{F(s)}$, where $y(t)$ satisfies the initial conditions $y(0^-) = y'(0^-) = 0$.

b. Find the impulse response of the oscillator, i.e. the solution of the initial value problem

$$y'' + 3y' + 2y = \delta(t)\ , \ \ y(0^-) = y'(0^-) = 0$$

c. Solve the initial value problem

$$y'' + 3y' + 2y = \frac{1}{1+e^t}\ , \ \ y(0) = 0\ , \ \ y'(0) = 0$$

by convolving the forcing term with the impulse response function.

d. Let r be an unspecified constant. Solve the initial value problem

$$y'' + 3y' + 2y = e^{rt}\ , \ \ y(0) = 1\ , \ \ y'(0) = 3,$$

by taking the Laplace transform of both sides of the equation. Verify that the solution is of the form

$$y = H(r)e^{rt} + Ae^{-t} + Be^{-2t}$$

where $H(s)$ is the transfer function and A and B are constants (depending of r).

e. Let ω be an unspecified constant. Use part **d** to show that the initial value problem

$$y'' + 3y' + 2y = \cos(\omega t)\ , \ \ y(0) = 1\ , \ \ y'(0) = 3$$

has a solution of the form

$$y = |H(i\omega)|\cos(\omega t - \phi) + Ce^{-t} + De^{-2t}.$$

where C, D, and ϕ are constants (all depending on ω). Therefore, $|H(i\omega)|$ is the frequency response function of the oscillator.

Exercise 4. The current $j(t)$ in a circuit with a resistor, capacitor, and inductor is modelled by an equation

$$\frac{d^2j}{dt^2} + 7\frac{dj}{dt} + 12j = \frac{dv}{dt}$$

where $v(t)$ is the voltage supplied to the circuit. The circuit is disconnected until time $t = 0$, so

$$j(t) = v(t) = 0 \text{ for } t < 0$$

and

$$j(0^-) = j'(0^-) = v(0^-) = 0$$

a. Let $J(s)$ and $V(s)$ be the Laplace transforms of $j(t)$ and $v(t)$. Find the ratio

$$H(s) = \frac{J(s)}{V(s)}.$$

In this context one refers to $H(s)$ as the transfer function of the circuit, because it is the ratio of the Laplace transform of the response (j) to the Laplace transform of the input (v).

b. Find the frequency response function of the oscillator, i.e. the amplitude of its steady state response to an alternating voltage $v(t) = \sin(\omega t)$ like the one provided by a wall outlet.

c. Suppose that at time $t = 0$ the circuit is connected to a 1-volt battery. Then

$$v(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Use the result in part **a** to solve for j, and plot it as a function of t. At what time does $j(t)$ attain its maximum value? Label it in your plot.

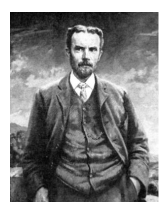
*Note: Technically the derivative of $v(t)$ is a delta function - it has an infinite value at $t = 0$. Using the result from part **a** allows you to avoid thinking about this delta function explicitly.*

d. Suppose the circuit is connected to the battery at time $t = 0$ and it is disconnected one second later. Then

$$v(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

Use the result in part **a** to solve for j. Give formulas for $j(t)$ on the intervals $0 < t < 1$ and $t > 1$, and plot j as a function of t. At what time does $j(t)$ attain its minimum value?

Pierre-Simon Laplace
(1749–1827)



Oliver Heaviside
(1850–1925)