HOMEWORK 7
MATH 18500-SECTION 41, 51
DUE: MAY 18TH

Exercise 1. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad \text{for} \quad 0 \leq x \leq 1\ ,\ \ t \geq 0, \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0.$$

These boundary conditions are called Neumann boundary conditions. You can think of the function $u(x,t)$ as modelling the temperature distribution in a metal rod of length 1 which is completely insulated from its surroundings.

a. Find all separated solutions which satisfy the given boundary conditions.

b. A general solution of the equation can be obtained by superimposing the separated solutions:

$$u(x,t) = \sum u_i(x,t) = \sum c_i X_i(x) T_i(t)$$

Show that any solution of this form also satisfies the given boundary conditions.

c. Find a cosine series for the function

$$f(x) = x$$

on the interval $[0,1]$, and use this to obtain a solution $u(x,t)$ which satisfies the initial condition

$$u(x,0) = f(x)$$

d. Evaluate the following limit:

$$\lim_{t\to\infty} u(x,t).$$

The result you obtain can be interpreted as follows: after a long time, the heat becomes uniformly distributed throughout the rod and the temperature is constant.

Exercise 2. Consider the heat equation

$$\frac{\partial u}{\partial t} = 7\frac{\partial^2 u}{\partial x^2} \quad (*)$$

with *inhomogeneous* Dirichlet boundary conditions,

$$u(0,t) = 1\ ,\ \ u(1,t) = 3 \quad (**)$$

You can think of the function $u(x,t)$ as modelling the temperature distribution of a rod which is insulated from its surroundings except at the two ends, where it is kept at two different constant temperatures.

a. A *steady state solution* of $(*)$ is a solution which does not change over time (i.e. $\frac{\partial u}{\partial t} = 0$) and therefore depends only on x:

$$u_{ss}(x,t) = u_{ss}(x)$$

Find a steady state solution which satisfies the boundary conditions $(**)$.

b. Let $u(x,t)$ be a solution of $(*)$ which satisfies the boundary conditions $(**)$. Show that the difference

$$u_h(x,t) = u(x,t) - u_{ss}(x)$$

is also solution, but satisfies the *homogeneous* Dirichlet boundary conditions

$$u_h(0,t) = u_h(L,t) = 0.$$

c. Assuming that $u(x,t)$ satisfies the initial condition

$$u(x,0) = x\ ,\ \ 0 < x < 1,$$

determine the initial value $u_h(x,0)$ and express it as a sine series. Use this to solve for $u_h(x,t)$.

d. Obtain the solution $u(x,t)$ by combining parts **b** and **c**, and show that

$$\lim_{t\to\infty} u(x,t) = u_{ss}(x)$$

i.e. the temperature distribution in the rod converges to the steady state distribution, as $t \to \infty$.

Exercise 3. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = 8\frac{\partial^2 u}{\partial x^2} - 2u$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$

You can think of the function $u(x,t)$ as modelling the temperature distribution in a metal rod which is insulated at its two ends, but not along its sides. In this scenario, heat is transferred from the sides of the rod to the surrounding air. In the equation we are using a temperature scale such that 0 is room temperature, and the factor $-2u$ tells us that the rod is is losing heat from its sides at a rate which is proportional to the temperature difference between the rod and the surrounding air (this is called Newton's law of cooling).

a. Find all separated solutions of the equation which satisfy the given boundary conditions.

b. Find a solution $u(x,t)$ which satisfies the initial condition

$$u(x,0) = x\ , \quad 0 < x < 1$$

c. Evaluate the limit

$$\lim_{t\to\infty} u(x,t)$$

and explain in terms of the physical model why this is a reasonable result.

For the following two problems we will be using the convention from this week's notes, where the Fourier transform of a function $f(x)$ is defined by

$$\mathcal{F}\left[f(x)\right] = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

With this convention, the Fourier inversion formula states that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk.$$

You should be aware that different authors use different conventions for the Fourier transform! In fact, we used different conventions in the notes and in the videos. Before consulting any reference which uses Fourier transforms, always check which convention it is using. Also, make sure when you apply the Fourier transform you are applying one convention consistently. For more information on different conventions:

https://physics.stackexchange.com/questions/308234/fourier-transform-standard-practice-for-physics

You will also encounter many different names for the frequency variable k, including p, E, ω, and ξ.

Exercise 4. Consider the function

$$\chi_{0,1}(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Compute the Fourier transform of $\chi_{0,1}(x)$.

b. In general, if $f(x)$ is a function which decays very rapidly as $x \to \pm\infty$, a is a constant, and

$$g(x) = f(x-a),$$

show that

$$\hat{g}(k) = e^{-ika}\hat{f}(k).$$

Hint: Write the Fourier transform of $f(x)$ as

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(u)e^{-iku}du$$

and make the substitution $u = x - a$.

c. In general, if $f(x)$ is a function which decays very rapidly as $x \to \pm\infty$, r is a positive constant, and

$$h(x) = f\left(rx\right),$$

show that

$$\hat{h}(k) = \frac{1}{r}\hat{f}\left(\frac{k}{r}\right)$$

*Hint: Proceed similarly to part **b** but make a different substitution.*

d. Combine parts **a**, **b**, and **c** to determine the Fourier transform of the function

$$\chi_{a,b}(x) = \begin{cases} 1 & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Then check your answer by computing the Fourier transform directly.

Exercise 5. *Optional.* Consider the Gaussian function

$$g(x) = e^{-x^2}.$$

a. In general, if $f(x)$ is any function which decays to 0 sufficiently rapidly as $x \to \pm\infty$, show that

$$\mathcal{F}\left[xf(x)\right] = i\frac{d}{dk}\left[\hat{f}(k)\right]$$

Hint: Expand $\hat{f}(k)$ using the definition of $\hat{f}$, and differentiate under the integral. The "sufficiently rapid" condition means that all integrals you write down converge - don't worry about it too much.

b. Show that $g(x)$ satisfies the differential equation

$$g'(x) + 2xg(x) = 0.$$

Then take the Fourier transform of both sides of this equation, applying part **a** and the standard rule for Fourier transforms of derivatives. The result will be a differential equation for $\hat{g}(k)$. Solve this equation, and obtain

$$\hat{g}(k) = Ce^{-\frac{k^2}{4}}$$

where C is a constant of integration.

c. On a problem set in math 184, you showed that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Assuming this fact, determine the value of C.
Hint: Evaluate $\hat{g}(0)$ using the definition of $\hat{g}$.

d. Verify that the Fourier inversion formula is valid for $g(x)$.
*Hint: From parts **b** and **c**, you have obtained (with different notation) the value of the integral*

$$\hat{g}(p) = \int_{-\infty}^{\infty} e^{-u^2} e^{-ipu} du$$

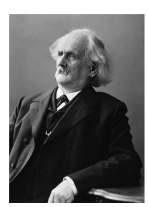
for any value of p. Make appropriate substitutions for p and u to obtain the value of the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(k) e^{-ikx} dk,$$

and conclude that it is equal to $g(x)$.

e. Verify that the Fourier inversion formula is valid for any function of the form $f(x)e^{-x^2}$, where $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ is a polynomial. This can be done using only the result in part **a**, and the rule for Fourier transforms of derivatives (applied to the inverse Fourier transform).

Johann Carl Friedrich Gauss
(1777–1855)



Carl Gottfried Neumann
(1832–1925)