

**HOMEWORK 6**  
**MATH 18500-SECTION 41, 51**  
**DUE: MAY 11TH**

**Exercise 1.** Let  $g(x)$  be the  $2\pi$ -periodic function which is equal to  $x$  on the interval  $(-\pi, \pi)$ , and let  $h(x)$  be the even  $2\pi$ -periodic function which is equal to  $\pi - x$  on the interval  $(0, \pi)$ .

- a. Graph the functions  $g(x)$  and  $h(x)$ .
- b. Find the complex Fourier series of  $g(x)$  by evaluating its complex Fourier integrals. Then convert the series to real form by applying Euler's formula and combining the  $e^{inx}$  and  $e^{-inx}$  terms.  
*Tip: The  $n = 0$  term is different from the others. That's OK, just write  $g(x) = \sum_{n \neq 0} c_n e^{inx}$ .*
- c. Find the real Fourier series of  $h(x)$  by evaluating its real Fourier integrals. Use symmetry to make your life easy - evaluate only one integral per coefficient. *Tip: The odd  $n$  coefficients are different from the even  $n$  coefficients. That's OK, just treat them separately when you write your series.*
- d. Find the real Fourier series of the odd  $2\pi$ -periodic function which is equal to  $\pi - x$  for  $x$  in  $(0, \pi)$ .  
*Hint: Do this without evaluating any integrals! Graph the function - how is it related to  $g(x)$ ?*

**Exercise 2.** Let  $f(t)$  be a function with an arbitrary period  $T$ . Then  $f(t)$  has a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \text{ where } \omega = \frac{2\pi}{T}$$

By imitating the derivation in the notes, show that

$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-ik\omega t} dt.$$

Also give formulas for  $a_n$  and  $b_n$  such that

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(\omega n t) + b_n \sin(\omega n t)$$

**Exercise 3.** Let  $f(x)$  be a continuous  $2\pi$ -periodic function, which is differentiable on the interval  $[-\pi, \pi]$  except possibly at finitely many points, and whose Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (*)$$

- a. By differentiating both sides of  $(*)$ , show that the Fourier coefficients of  $f'$  and  $f$  are related by

$$c'_n = inc_n$$

Confirm this by applying the formula for the coefficients,

$$c'_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx$$

and integrating by parts on the right hand side. Use the fact that  $f(x)$  is periodic, so  $f(\pi) = f(-\pi)$ . *Technical point: When you integrate by parts, you are applying the fundamental theorem of calculus. Typically this theorem is only stated for differentiable functions, but it is also valid for continuous functions which are differentiable except at finitely many points.*

- b. Graph the derivative of the function  $h(x)$  from Problem 1, and compute its Fourier series using **a**. Confirm your answer by directly computing the coefficients using complex Fourier integrals.
- c. For the rest of the problem, let  $f(x)$  be the  $2\pi$ -periodic function which is equal to  $\frac{x^2}{2}$  for  $x$  in  $[-\pi, \pi]$ . Graph  $f(x)$ , observe that it satisfies the hypotheses required for **a**, and notice that its derivative is the function

$g(x)$  from problem 1. Then use  $\mathbf{a}$  to determine the Fourier coefficients  $c_n$  of  $f(x)$  with  $n \neq 0$ . Determine  $c_0$  using the usual method. Finally, convert the Fourier series to real form and obtain

$$\frac{x^2}{2} = \frac{\pi^2}{6} + 2 \cdot \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{n^2}, \quad \text{for } -\pi \leq x \leq \pi$$

- d. Set  $x = \pi$  in the formula above and deduce the value of the infinite sum  $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ .

**Exercise 4.** Consider the boundary value problem

$$y'' + 4y = 4x, \quad y(0) = y(1) = 0$$

- a. Find the Fourier series of the function  $4x$  on the interval  $[-1, 1]$ , either by making a substitution in the series you found in **1b** or using the formula in **2**. You should obtain an equation of the form

$$4x = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

- b. Solve the boundary problem, by solving the individual equations

$$y_n'' + 4y_n = \sin(n\pi x), \quad y_n(0) = y_n(1) = 0$$

and applying the superposition principle.

- c. Solve the boundary value problem directly, using the method of undetermined coefficients. Then compute the sine series of the solution, and show that it is consistent with your answer to part **b**.

**Exercise 5.** *Optional.* You are pushing a small child on a swing. You're trying not to push too hard, so the swing stays close to its vertical equilibrium position and its motion can be modeled by a linear equation

$$mL\theta'' + 2b\theta' + mg\theta = f(t).$$

Here  $\theta(t)$  is the angular displacement of the swing,  $f(t)$  is the force you exert on the swing,  $m$  is the mass of the child,  $L$  is the length of the swing,  $g$  is the gravitational constant, and  $b$  is a damping coefficient determined by various physical factors (air resistance, rust on the swing, etc.).

If you wish, you may substitute the following realistic values:  $m = 12$  kg,  $b = 1 \frac{\text{kg}\cdot\text{m}}{\text{s}}$ ,  $g = 10 \frac{\text{m}}{\text{s}^2}$ ,  $L = 2.5$  m. You may do so at any point during the problem. However, it is nicer to work everything out in general.

- a. You give the swing pushes at regular intervals, repeating with period  $T$ . For  $-\frac{T}{2} \leq t \leq \frac{T}{2}$  the force you exert is given by

$$f(t) = \begin{cases} F & -\frac{\Delta t}{2} < t < \frac{\Delta t}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Here  $F$  is the force you push with, and  $\Delta t$  is the duration of the pushes. Graph the function  $f(t)$ .

- b. Write  $f(t)$  as a Fourier series,

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t), \quad \text{where } \omega = \frac{2\pi}{T}.$$

Explain in words why all the coefficients  $b_n$  are equal to 0, and calculate the values of  $c_0$  and  $a_n$ .

- c. The steady state motion of the swing can be written in the form

$$\theta(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t - \phi_n)$$

Determine  $C_0$  and  $A_n$  explicitly (you do not have to determine  $\phi_n$ ).

- d. Assuming that the pushes have a very short duration, use the approximation

$$\sin(x) \approx x \quad \text{for small values of } x$$

to show that

$$A_n \approx \frac{1}{\pi} \frac{F\Delta t}{\sqrt{\left(\frac{mg}{\omega} - mLn^2\omega\right)^2 + (2bn)^2}}.$$

Use this approximation for the rest of the problem.

- e. Find the value of  $T$  which maximizes the value of  $A_1$ . Note that this is equal to the *natural period* of the swing (the period with which the swing would oscillate in the absence of damping and forcing)! Therefore, to give the child maximum enjoyment, you must push the swing at its natural frequency.
- f. Say you are lazy and push at *half* the natural frequency. In this case, substitute the values of  $m$ ,  $L$ ,  $g$ , and  $b$  given at the beginning of this problem and calculate the first few amplitudes  $A_n$ . Which one is largest and by what factor? Explain why this makes sense, given your experience with swings.



Jean Baptiste Joseph Fourier  
(1758–1830)



Johann Peter Gustav Lejeune Dirichlet  
(1805–1859)

