HOMEWORK 5
MATH 18500-SECTION 41, 51
DUE: WEDNESDAY MAY 4TH

Exercise 1. Find the general solution of each of the following second order inhomogeneous equations:

a. $y'' + 3y' + 2y = e^{2t}$
b. $y'' + 3y' + 2y = e^{-2t}$
c. $y'' + 4y' + 4y = e^{-2t}$

Exercise 2. Find a particular solution of the equation

$$y'' + 3y' + 2y = \cos t + 2\sin t$$

and write it in each of the following forms:

a. $y = \text{Re}\left[Ze^{i\omega t}\right]$ where $\omega > 0$ and Z is a complex number.
b. $y = a\cos(\omega t) + b\sin(\omega t)$ where $\omega > 0$ and a and b are real numbers.
c. $y = A\cos(\omega t - \phi)$ where $A > 0$, $\omega > 0$, and $0 \leq \phi < 2\pi$.

Exercise 3. Solve the following initial value problem:

$$y'' + 3y' + 2y = 5e^{2t} + 6e^{-2t} + 3\cos(t) + 6\sin(t)\ ,\quad y(0) = 1\ ,\quad y'(0) = 1$$

Hint: Use the work you did in problems 1 and 2!

Exercise 4. Consider a general second order equation of the form

$$(D - \lambda_1)(D - \lambda_2)y = e^{\lambda_1 t}$$

where λ_1 and λ_2 are arbitrary constants.

a. If $\lambda_1 \neq \lambda_2$, the method of undetermined coefficients says to guess a particular solution of the form

$$y = Ate^{\lambda_1 t}.$$

for some value of A. This leads to a general solution of the form

$$y = Ate^{\lambda_1 t} + Be^{\lambda_1 t} + Ce^{\lambda_2 t}.$$

Justify this "lucky guess" using the repeated integration method.

b. If $\lambda_1 = \lambda_2$, the method of undetermined coefficients says to guess a particular solution of the form

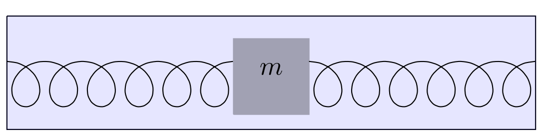
$$y = At^2 e^{\lambda_1 t}$$

for some value of A. This leads to a general solution of the form

$$y = At^2 e^{\lambda_1 t} + Bte^{\lambda_1 t} + Ce^{\lambda_1 t}.$$

Justify this "lucky guess" using the repeated integration method.

Exercise 5. Consider a fluid-filled box which contains a solid block attached to the ends of the box with two identical springs:

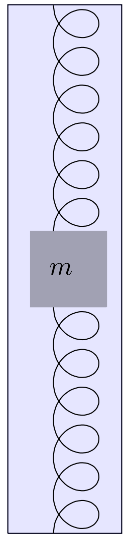


Let y denote the displacement of the block from the center of the box (so in the picture above, $y = 0$). Then the motion of the block can be modelled using an equation of the form

$$my'' + ly' + ky = f(t),$$

where $m > 0$ represents the mass of the block, $k > 0$ represents the tightness of the spring, $l > 0$ represents the stickiness (*viscosity*) of the fluid, and $f(t)$ represents the sum of all external forces acting on the block. Assume that the mass of the block is 1 kilogram.

a. You suddenly turn the box on its side, like this:



Now block is subject to the force of gravity, and its motion can be modeled by solving the initial value problem

$$my'' + ly' + ky = -mg \ , \quad y(0) = 0 \ , \quad y'(0) = 0,$$

where $g \approx 10$ meters/second^2. If the block comes to rest 10 centimeters below its original equilibrium position, determine the value of k (in units of kilograms/second^2).

b. As the block comes to rest, it oscillates around its new equilibrium position (10 centimeters below its original equilibrium position). Using a high speed camera, you measure the times at which the object passes through its new equilibrium position, and find this is happenening once per second. Based on this experiment, and your answer to part **a**, determine the value of l (in units of kilograms/second).

Exercise 6. *Optional.* Consider a similar box to the one described in problem 5, but assume the values $m = k = 1$. Instead of turning the box on its side, you gently rock it back and forth at a constant frequency. Now the motion of the block can be modeled by solving the initial value problem

$$y'' + ly' + y = A_d \cos(\omega_d t) \ , \quad y(0) = 0 \ , \quad y'(0) = 0.$$

where ω_d and A_d are positive constants (the *driving frequency* and *driving amplitude*).

a. Show that the solution takes the form

$$y = A\cos(\omega_d t - \phi) + y_h(t)$$

where A and ϕ are positive constants and $y_h(t)$ is a solution of the homogeneous equation

$$y'' + ly' + y = 0.$$

Give explicit formulas for A and ϕ in terms of A_d, ω_d, and l.

b. Note that $\lim_{t\to\infty} y_h(t) = 0$ for any value of l. Because of this, $y_h(t)$ is often referred to as a *transient*. As you increase the value of l, does the transient die off more or less rapidly?

c. The ratio A/A_d is called the *amplitude gain* of the system. Using a computer, plot the amplitude gain as a function of ω, for a few different values of l:

$$l = 0.01, 0.1, 1, 10, 100$$

d. Let ω_r be the value of the driving frequency which results in the maximum amplitude gain (the *resonant frequency*). Give a formula for ω_r as a function of l, assuming that l is small and positive. Is it greater or less than the natural frequency ω_0? (*See problem set 4 for the definition of* ω_0)

e. If the value of l is sufficiently large, then the resonant frequency will be 0. Give a formula for the smallest value of l such that this occurs. Is it greater than, less than, or equal to the value of l at which critical damping occurs?

f. What is the solution of the initial value problem if $l = 0$ and $\omega_d = \omega_r$? Plot it as a function of t.

Exercise 7. *Optional*. The goal of this problem is to obtain a particular solution of the equation

$$(D-\lambda_1)(D-\lambda_2)y = f(t),$$

where λ_1 and λ_2 are arbitrary constants, and $f(t)$ is an arbitrary function (not a sum of exponentials).

a. Let λ be an arbitrary constant. Use the integrating factor method to show that the solution of the initial value problem

$$(D-\lambda)x = f(t)\ ,\quad x(0)=0$$

is given by the integral

$$x(t) = e^{\lambda t}\int_0^t f(u)e^{-\lambda u}du = \int_0^t f(u)e^{\lambda(t-u)}du.$$

Hint: When you get to the integration step of the integrating factors method, integrate both sides of the equation from 0 *to* t *instead of doing an indefinite integral. Since the variable* t *appears in the upper limit of your integral, you must rename the variable inside the integral to avoid confusion. If you call that variable* u *instead of* t*, you will obtain the formula above.*

b. Let λ_1 and λ_2 be arbitrary constants. Use the repeated integration method to show that the solution of the initial value problem

$$(D-\lambda_1)(D-\lambda_2)y = f(t)\ ,\quad y(0)=0\ ,\quad y'(0)=0$$

is given by the iterated integral

$$y(t) = \int_{v=0}^t\int_{u=0}^v f(u)e^{\lambda_1(v-u)}e^{\lambda_2(t-v)}dudv.$$

Hint: Apply the repeated integration approach. In each step, use the formula from part ***a****. To avoid confusion, call your integration variable* v *instead of* u *the second time you integrate.*

c. Sketch the region of integration for the double integral in part **b**, and set up the same integral with the opposite order of integration ($dvdu$ instead of $dudv$).

d. Assuming that $\lambda_1 \neq \lambda_2$, evaluate the inner integral. You should obtain the following formula:

$$y(t) = \int_0^t f(u)\left(\frac{e^{\lambda_1(t-u)} - e^{\lambda_2(t-u)}}{\lambda_1-\lambda_2}\right)du = \int_0^t f(u)g(t-u)du.$$

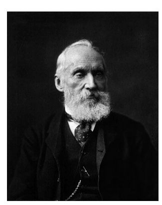
Observe that the function $g(t-u)$ is a solution of the initial value problem

$$(D-\lambda_1)(D-\lambda_2)y = 0\ ,\quad y(u)=0\ ,\quad y'(u)=1.$$

e. If $\lambda_1 = p+iq$ and $\lambda_2 = p-iq$ are complex conjugates, use Euler's formula to simplify the integral in part **d** to an integral which does not involve any complex numbers.

f. Generalize parts **d** and **e** to the case $\lambda_1=\lambda_2$.

Thomas Young
(1773–1829)



William Thomson, 1st Baron Kelvin
(1824–1907)

Hermann von Helmholtz
(1821–1894)