Homework 1
Math 18500-Section 41, 51

Due: Wednesday April 6th

Exercise 1. For each differential equation below:

1. Sketch the isoclines with slopes -2, -1, 0, 1, and 2.
   Tip: Use dashed/dotted lines for isoclines, or a lighter color.
2. Use the isoclines you drew to sketch the slope field.
3. Sketch three solutions, with initial values $y(0) > 0$, $y(0) = 0$, and $y(0) < 0$. *Tip: Use solid lines or a darker color for the solutions.*
4. Use your sketch to identify an explicit solution, and check by substituting in the equation.
5. If possible, identify the asymptotic behavior of the solutions as $x \to \pm\infty$.
   In other words, identify the asymptotes of the solutions.

It's a good idea to use computer software like geogebra or grapher to confirm your sketch is accurate. However, you must *create* your sketch using the isocline method. Sketches that were clearly produced by computer (i.e. with evenly spaced slope lines, instead of slope lines along isoclines) will not receive credit. Also beware - numerical solutions produced by computer may be inaccurate in some cases. Think about what you are doing, don't let the computer think for you!

a. $\frac{dy}{dx} = y - x + 1$.

b. $\frac{dy}{dx} = xy$.

c. $\frac{dy}{dx} = x^2 - y^2 + 1$

Tips for making sketches:

1. *Be very careful to draw the same slope at every point on the isocline.*
2. *Solutions have local maxima/minima or inflection points on the isocline of slope* 0.
3. *Look out for situations where the solution gets "trapped" between two isoclines.*

Exercise 2. For each autonomous equation, sketch several solutions which exhibit all possible behaviors. Identify any equilibrium solutions, and determine whether they are stable/unstable/semistable.

a. $y' = \sin(y)$
Do not solve this equation!

b. $y' = (y-1)(y-2)(y-3)$
Do not solve this equation!

c. $y' = y^2(y-1)(y+1)$
Do not solve this equation!

Exercise 3. For each differential equation, find all of the solutions, and determine which of these solutions has the given intial value:

a. $y' + 1 = 2y, \quad y(0) = 1$

b. $y' = 6e^{2x-y}, \;\; y(1) = 2.$

c. $y' = 1 + x + y + xy, \quad y(-1) = 2.$

Hint: In each case it is possible to separate the variables.

Exercise 4. Let $y' = f(y)$ be an autonomous equation with **precisely** two equilibrium values y_1 and y_2. Assume that $f(y)$ is a continuous function with a continuous derivative, defined for all real values of y.

a. Is it possible for both y_1 and y_2 to be stable?

b. Is it possible for both y_1 and y_2 to be unstable?
(*Here we consider "semistable" equilibria to be unstable, since they are unstable in one direction*).

If your answer is "yes", give a specific example. If your answer is "no", give an explanation.

Exercise 5. A circuit with a resistor, capacitor, and battery in series can be modeled by the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance of the resistor, C is the capacitance of the capacitor, $Q = Q(t)$ is the charge built up on the capacitor, and V is the voltage supplied by the battery.
Note: It's OK if you don't know what a capacitor is. Just think of R, C, and V as arbitrary positive constants, and think of Q as an unknown function of t.

a. Suppose the capacitor is completely discharged, and at time $t = 0$ we connect the circuit to a battery (so, $Q(0) = 0$). How much charge eventually builds up on the capacitor, in the limit as $t \to \infty$?
Hint: It is not necessary to solve the equation. Express your answer in terms of R, C, and/or V.

b. Solve for Q as a function of t, and confirm your answer to part **a** by computing its limit as $t \to \infty$.
Your answer will depend on R, C, and/or V.

c. How much time is required for the capacitor to become halfway charged?
Your answer will depend on R, C, and/or V.

d. If the capacitance was increased, would more or less time be required?

Tip for those who are uncomfortable with "less variable variables": if you find it confusing to have named constants like R, C, V, replace them with easily recognizable numbers (e.g. $\cos(17)$, π, $\sqrt{11}$*). Then do the problem in pencil. When you're done, erase all of the numbers and put in the corresponding variable names.*

Exercise 6. *Optional*. Consider a chemical equilibrium

$$X + Y \leftrightarrows Z,$$

where X, Y, and Z are chemical compounds whose concentrations are given as functions of time by $x(t)$, $y(t)$, and $z(t)$. The concentration of Z can be modeled by the differential equation

$$\frac{dz}{dt} = k_+xy - k_-z$$

where k_+ and k_- are constants (the *rate constants* of the forward and backward reactions). For simplicity, take $k_+ = k_- = 1$, so that

$$\frac{dz}{dt} = xy - z.$$

a. Explain physically why we must have

$$x - x_0 = y - y_0 = z_0 - z$$

where $x(0) = x_0$, $y(0) = y_0$, and $z(0) = z_0$ are the initial concentrations.

b. Assuming part **a**, show that $z(t)$ satisfies a first order differential equation of the form

$$\frac{dz}{dt} = a + bz + cz^2,$$

where a, b, and c are constants depending on x_0, y_0, and z_0.

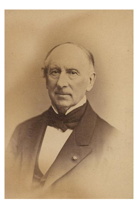
c. Assume for simplicity that the initial concentrations are $x_0 = 3$, $y_0 = 4$, $z_0 = 0$. What will be the limiting concentration of Z,

$$\lim_{t\to\infty} z(t)?$$

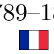
Hint: It is not necessary to solve the equation.

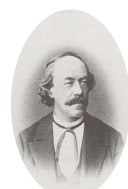
d. Suppose that the reaction rates were instead given by $k_+ = 1$ and $k_- = 6$. Explain, from a physical point of view, why the limiting concentration of Z would be lower in this case. Verify this mathematically, by computing the limiting concentration exactly.

e. Verify your answer to part **c**, by solving for $z(t)$ and computing the limit.



Augustin Louis Cauchy
(1789–1857)





Rudolf Otto Sigismund Lipschitz
(1832–1903)