Homework 2
Math 18400-Section 41, 51

Due: Monday January 24th

Exercise 1. Using the multivariable chain rule,

1. Write $g'(x)$ in terms of the partial derivatives of f for $g(x) = f(x, x)$.
2. Write $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ in terms of the partial derivatives of f for $g(x, y) = f(y, f(x, x))$.
3. Write $g'(t)$ in terms of the partial derivatives of f for $g(t) = f(2 + 2t, t^2)$.
4. Write $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$ in terms of the partial derivatives of f for $g(u, v) = f(uv, u^2 + v^2)$.
5. Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $y = \sin(x - t)$ and $x = te^{2t}$, at the point where $x = 0$.
6. Compute $\dfrac{\partial z}{\partial y}$, if $x = 3u - u^3 + 3v^2u$, $y = 3v - v^3 + 3u^2v$, and $z = 3u^2 - 3v^2$, at the point where $u = v = 1$.

Exercise 2. Any object conducts heat - if we heat up one part of the object (i.e. increase its temperature), then eventually that heat will spread throughout the object. Fourier's law of heat conduction states that if $T(x, y, z)$ gives the temperature of the object at the location (x, y, z), then heat flows in the direction

$$\vec{q} = -\vec{\nabla}T.$$

In this problem we will consider a 2D metal sheet with temperature function $T(x, y) = xy - x$.

1. At the point $(1, 1)$, in which direction does the temperature increase the fastest (per unit distance)?
2. How fast does the temperature increase (per unit distance) in the direction of the vector $3\hat{\imath} + 4\hat{\jmath}$?
3. Sketch the *isothermal curves* $T = -2, -1, 0, 1, 2$. At several points along each isothermal curve, including the point $(1, 1)$, draw a vector which indicates the direction that heat would flow.
4. Sketch the curves along which heat would flow (i.e. curves tangent to the direction of heat flow)

Exercise 3. Let $\vec{x}(t)$ be a parameterized curve which describes the motion of a satellite orbiting a planet. In this situation, $\vec{x}(t)$ obeys Newton's law of gravity, which states that

$$m\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}U,$$

where

$$U = -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}}$$

is the gravitational potential energy of the satellite. Here G is a constant, M is the mass of the planet, and m is the mass of the satellite.

1. Compute the gradient of U, and show that its magnitude is given by

$$|\vec{\nabla}U| = \frac{GMm}{x^2 + y^2 + z^2}.$$

2. Deduce from Newton's law that the angular momentum of the satellite,

$$\vec{L} = m\vec{x} \times \frac{d\vec{x}}{dt}$$

is constant. Conclude that the satellite stays in a plane through the origin, with normal vector $\vec{L}$. (*Hint: Use the product rule for cross products, which can be found on page 286 of your textbook.*)

Exercise 4. Consider the following parameterized curve, which represents the motion of a charged particle:

$$\vec{r}(t) = (\cos 3t, -4t, \sin 3t),$$

1. Sketch the trajectory of $\vec{r}(t)$. Indicate the direction of motion in your sketch.

2. Newton's law for a charged particle moving in a magnetic field states that

$$m\vec{a} = q\vec{v} \times \vec{B}$$

where m is the mass of the particle, q is the charge on the particle, $\vec{B}$ is the magnetic field, $\vec{v}$ is the velocity of the particle, and $\vec{a}$ is the acceleration of the particle. Verify that $\vec{r}(t)$ satisfies Newton's law, if $q = 3m$ and $\vec{B} = \vec{j}$.

3. Show that the velocity and acceleration of $\vec{r}(t)$ are perpendicular, and show that its speed is constant.

4. In general, show that *every* parameterized curve whose velocity and acceleration are perpendicular has constant speed. *Hint: A function is constant if and only if its derivative is* 0.

5. Conclude that *any* charged particle in a magnetic field travels at constant speed.

Exercise 5. We say that f is homogeneous of degree r if

$$\text{For all } t > 0,\ f(tx, ty) = t^r f(x, y).$$

1. Give an example of an homogeneous function of degree 3.

2. Show that if f is homogeneous of degree r then its partial derivatives are homogeneous of degree $r - 1$.

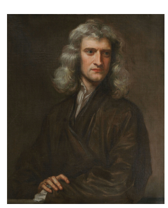
3. By differentiating the definition with respect to t and using **2.**, show that if f is homogeneous of degree r then

$$x\partial_x f(x, y) + y\partial_y f(x, y) = r f(x, y).$$

4. Show that we have

$$x^2\partial_x^2 f(x, y) + 2xy\partial_{xy}^2 f(x, y) + y^2\partial_y^2 f(x, y) = r(r-1) f(x, y).$$

Jean-Baptiste Joseph Fourier
(1768–1830)



Isaac Newton
(1643–1727)

Leonhard Euler
(1707–1783)