Homework 1
Math 18400-Section 41, 51

Due: Monday January 17th

Exercise 1. Consider the function

$$f(x, y) = -x^2 + 2y^2 - y^4.$$

1. Sketch the curves obtained by slicing the graph of $f(x, y)$ with the planes $y = -1, 0, 1$, and $x = 0$.
2. Use your work from part **a** to sketch the graph of $f(x, y)$.
3. In the sketch you made in part **b**, also sketch the level curves $z = -2, -1, 0, 1, 2$.
4. Use your work in part **c** to sketch the contour plot of $f(x, y)$.
5. At which points is the tangent plane to the graph horizontal? Justify using partial derivatives.
6. Near each point where the tangent plane is horizontal, determine (based on your sketches) whether the graph of $f(x, y)$ lies entirely above its tangent plane, entirely below its tangent plane, or neither.

Exercise 2. Compute all partial derivatives of the first and second order for the following functions:

1. $f(x, y) = 3^{x/y}$	**2.** $f(x, y) = \cos(x^2 + y)$
3. $f(x, y) = \arctan\left(\frac{y}{x^2}\right)$	**4.** $f(x, y, z) = y\sin(xz)$

Exercise 3. Describe the domain and find the equation of the tangent plane at the given point (x_0, y_0, z_0) for each function below

1. $f(x, y) = \sqrt{19 - x^2 - y^2}$ at $(x_0, y_0, z_0) = (1, 3, 3)$.
2. $g(x, y) = \sin(\pi xy)\mathrm{e}^{2x^2y-1}$ at $(x_0, y_0, z_0) = (1, \frac{1}{2}, 1)$.
3. $h(x, y) = x^2 - 2y^3$ at any point $(2, 1, 2)$.
4. Find all points $M_0 = (x_0, y_0, h(x_0, y_0))$ on the graph of h such that the tangent plane at M_0 is parallel to the tangent plane at $(2, 1, 2)$ (the plane you find in **3.**).

Exercise 4. **1.** Use a (well-chosen) quadratic approximation to compute, up to a small error,

$$\mathrm{e}^{\sin(3.16)\cos(0.02)}.$$

2. Use an affine approximation to compute, up to a small error,

$$\arctan\left(\sqrt{4.03} - 2\mathrm{e}^{0.01}\right)$$

Exercise 5. The **Laplacian** of a function $h(x, y, z)$ is defined as follows:

$$\vec{\nabla}^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}$$

We say that h is **harmonic** if its Laplacian is equal to 0:

$$\vec{\nabla}^2 h = 0$$

1. The gravitational potential energy due to a point mass at the origin is given by

$$U(x, y, z) = \frac{k}{\sqrt{x^2 + y^2 + z^2}}.$$

where k is a constant. Show that U is a harmonic function.

2. The electric potential energy due to an infinite uniform line of charge along the z axis is given by

$$V(x, y, z) = k \ln \left(\frac{1}{\sqrt{x^2 + y^2}} \right)$$

where k is a constant. Show that V is a harmonic function.

3. Suppose that $u(x, y)$ and $v(x, y)$ are functions satisfying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Show that both $u(x, y)$ and $v(x, y)$ are harmonic.
In a situation like this, we say that u and v are ***harmonic conjugates****.*

4. Show that the functions $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ are harmonic conjugates (and therefore, they are harmonic).

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