Homework 7
Math 18400-Section 41, 51

Due: Wednesday March 2nd

Exercise 1. Compute the following fluxes in two different ways: directly from the definition of flux, and by applying the divergence theorem.

a. The inward flux of the 2D vector field $x^2\ \hat{\imath} + y^2\ \hat{\jmath}$ through the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

b. The outward flux of the 3D vector field $(x^2+y^2+z^2)(x\hat{\imath}+y\hat{\jmath}+z\hat{k})$ through a sphere of radius 5 centered at the origin.

c. The flux of the $3D$ vector field $\vec{F} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ through the parallelogram with vertices $(1,1,1)$, $(2,1,2)$, $(1,2,2)$, and $(2,2,3)$.

Hint: Make your life easy by using geometric reasoning when possible.

Exercise 2. Consider the vector field $\vec{F} = x\hat{\imath} + y\hat{\jmath}$.

a. Draw a sketch of $\vec{F}$ which accurately depicts its magnitude and direction at a large number of points.

b. Show that the flux of $\vec{F}$ through any closed curve is two times the area enclosed by that curve.
Hint: Apply the divergence theorem.

c. Use part **b** to find the total area enclosed by the following ellipse:

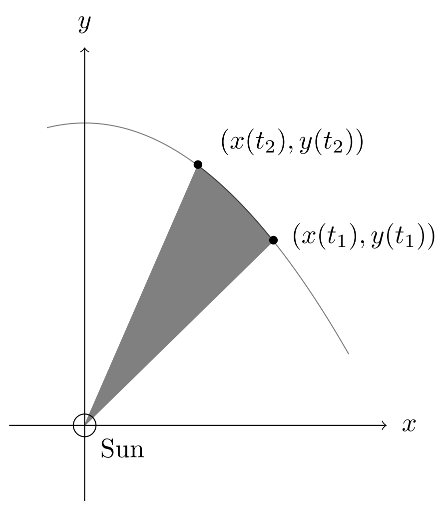
$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

You must parameterize the ellipse and explicitly compute the flux of $\vec{F}$ through it.

d. Check your answer to **c** by using the change of variables $x = 3u$, $y = 4v$ to evaluate $\iint_R dA$.

e. In general, what is the area enclosed by an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

f. One of Kepler's laws of planetary motion states that as a planet moves, it sweeps out equal areas in equal times. The area swept out between two times can be visualized using the following diagram:



Show that Kepler's law is a consequence of the fact that the angular momentum of the planet,

$$L = x(t)y'(t) - y(t)x'(t)$$

is constant. *Hint: Use part **b** to compute the area swept out between two times t_1 and t_2, and show that it only depends on $\Delta t = t_2 - t_1$. Make your life easy by using geometric reasoning when possible.*

Exercise 3. Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ which lies above the xy plane, and consider the vector field $\vec{G} = x\hat{i} + y\hat{j} + z\hat{k}$. Compute the flux of $\vec{G}$ through S in three different ways:

a. By parameterizing S using cylindrical coordinates and directly applying the definition of flux.

b. By parameterizing S using rectangular coordinates and directly applying the definition of flux.

c. By applying the divergence theorem to a region B which is bounded by S and another surface S'.

Johannes Kepler
(1571–1630)

James Clerk Maxwell
(1831–1879)

Exercise 4. *(Optional for extra credit).* The *electric* and *magnetic* fields are important examples of vector fields from physics. In electrostatics, the electric field is time-independent, and can be written in the form

$$\vec{E} = -\vec{\nabla}\phi$$

where $\phi = \phi(x, y, z)$ is a function of 3 variables, called the *electric potential.* The electric potential satisfies *Poisson's equation*,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -4\pi\rho,$$

where $\rho(x, y, z)$ is the charge density function (net charge per unit volume).
In magnetostatics, the magnetic field $\vec{B}$ is time-independent, and can be written in the form

$$\vec{B} = \text{curl}(\vec{A}) = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

where $\vec{A} = A_1(x, y, z)\hat{\imath} + A_2(x, y, z)\hat{\jmath} + A_3(x, y, z)\hat{k}$ is another vector field, called the *magnetic vector potential.*

a. Given any surface S, the *magnetic flux* through S is defined by

$$\text{Magnetic Flux} = \iint_S \vec{B} \cdot \hat{N} dA$$

Show that the magnetic flux through any closed surface S is equal to 0.
Don't use other things you know about magnetic fields, only use the assumption $\vec{B} = \text{curl}(\vec{A})$.

b. Given any surface S, the *electric flux* through S is defined by

$$\text{Electric Flux} = \iint_S \vec{E} \cdot \hat{N} dA$$

Show that the electric flux through any closed surface S is equal to $4\pi Q$, where Q is the net charge in the region enclosed by S.
Don't use other things you know about electric fields, only use the assumptions stated in this problem.

c. Suppose that the electric potential is given by

$$\phi(x, y, z) = \frac{\vec{p} \cdot \vec{x}}{r^3} = \frac{p_1 x + p_2 y + p_3 z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

where $\vec{p} = p_1\hat{\imath} + p_2\hat{\jmath} + p_3\hat{k}$ is an arbitrary but constant vector. Determine the electric flux through a sphere of radius R centered at the origin.

d. Explain why it would not have been legitimate to apply the divergence theorem in part **c**.

e. Take a look at the following wikipedia article,

`https://en.wikipedia.org/wiki/Dipole`,

particularly the sections titled "Classification" and "Field from an electric dipole". Then explain why your answer in part **c** makes sense physically.