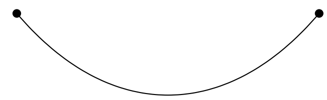
Homework 6
Math 18400-Section 41, 51

Due: Wednesday February 23th

Exercise 1. **a**. Determine the length of the *catenary* curve

$$y = \frac{e^x + e^{-x}}{2}\ ,\quad -1 \leq x \leq 1$$

This is the shape that a string makes when suspended between two points:



Hint: For this problem it is helpful to introduce the hyperbolic sine and cosine functions

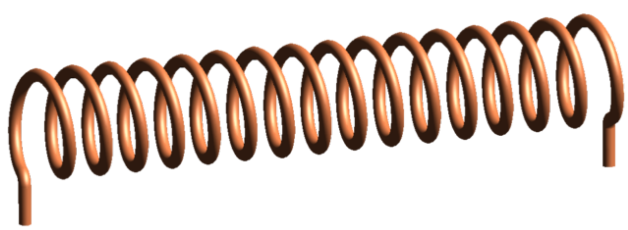
$$\sinh(x) = \frac{e^x - e^{-x}}{2}\ ,\quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

These functions satisfy the following identities (which you should verify, if you want to use them):

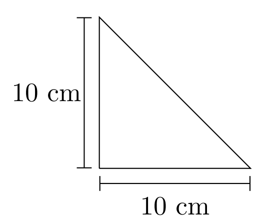
$$\frac{d}{dx}\cosh(x) = \sinh(x)\ ,\quad \frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\cosh^2(x) = 1 + \sinh^2(x).$$

b. You have a spool of copper wire which has a linear density of 1g/cm. You want to make a solenoid by wrapping the wire 100 times around a cylinder which is 10cm tall and 4cm in diameter. How many linear centimeters of wire do you need to make the solenoid? *Assume that the width of the wire is negligible and the coils are evenly spaced. For reference, here is what a solenoid looks like:*



c. You bend the same copper wire from part **b** into a right isoceles triangle with side length 10 cm:



Determine the moment of inertia of the triangle around an axis perpendicular to its hypotenuse and passing through its center of mass. *Note: This problem requires line integrals, not a double integral over the region bounded by the triangle.*

Exercise 2. For each of the following surfaces, use the specified coordinate systems to parameterize the surface. You must clearly sketch the domain of each parameterization.

a. The part of the paraboloid $z = x^2 + y^2$ which lies in the first octant($x \geq 0$,$y \geq 0$, $z \geq 0$) and below the plane $z = 4$. *First use rectangular coordinates* (x, y)*, then use cylindrical coordinates* (ρ, ϕ)*.*

b. The surface which is cut out from the cone $3x^2 + 3y^2 = z^2$, $z > 0$, by the cylinder $x^2 + (y - 1)^2 = 1$.
First use cylindrical coordinates (z, ϕ)*, then use spherical coordinates* (r, ϕ)*.*

c. The part of the sphere $x^2 + y^2 + z^2 = 9$ which is cut out by the inequalities $x \leq y$ and $z \leq 0$.
First use cylindrical coordinates (z, ϕ)*, then use spherical coordinates* (ϕ, θ)*.*

Exercise 3. For each of the following surfaces, calculate its surface area:

a. The surface in part **a** of Problem 2.

b. The surface in part **b** of Problem 2.

c. The portion of a sphere of radius R which is between two parallel planes separated by a distance D.
It turns out that this area does not depend on how close the planes are to the center of the sphere! You shouldn't assume this - it will be an interesting consequence of your calculations. However, you may assume that the sphere is centered at the origin and the planes are both horizontal.

d. The *catenoid* surface,

$$x^2 + z^2 = \left(\frac{e^y + e^{-y}}{2}\right)^2 \ , \quad -1 \leq y \leq 1$$

This is the shape formed by a soap film suspended between two rings of equal size:

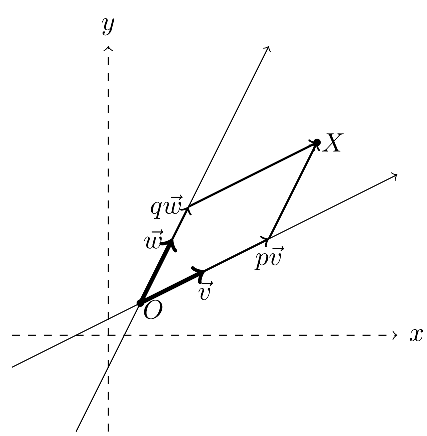
`https://www.youtube.com/watch?v=GFGKKwQHb3Q`

It can be shown that the catenoid has the least surface area, of all surfaces joining the two rings.

Exercise 4. Calculate the Jacobian factor for each of the following coordinate systems:

a. Spherical coordinates in three dimensions.

b. A linear coordinate system (p, q) in 2 dimensions, which is defined by the following diagram:



Here the origin of the coordinate system is the point $O = (1, 1)$, its basis vectors are

$$\vec{v} = \hat{ß} + 2\hat{æ} \ , \quad \vec{w} = 2\hat{ß} + \hat{æ},$$

and the (p, q) coordinates of a point $X = (x, y)$ are defined by the vector equation

$$\overrightarrow{OX} = p\vec{v} + q\vec{w}.$$

Exercise 5. Consider the change of variables $u = x^2 - y^2$, $v = x + y$.

a. Calculate the Jacobian factor $\frac{\partial(u,v)}{\partial(x,y)}$.

b. Solve for x and y in terms of u and v, and use this to calculate the inverse Jacobian factor $\frac{\partial(x,y)}{\partial(u,v)}$.

c. In general, show that if $x = x(u, v)$, $y = y(u, v)$ is a change of variables, and $u = u(x, y)$, $v = v(x, y))$ is the inverse change of variables, then

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1.$$

Hint: First use the chain rule to show that the Jacobian matrices are inverses of each other:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Then apply the following matrix identity, which is valid for arbitrary $n \times n$ matrices A and B:

$$\det AB = \det A \det B.$$

It's an interesting (but optional) exercise to verify that this identity holds for 2×2 matrices.

d. Verify that the result in **c** is consistent with your computations from parts **a** and **b**.

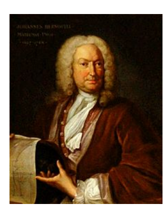
Exercise 6. Use the change of variables formula to calculate the following quantities:

a. The area of the region bounded by the following four curves: the line $x + y = 1$, the line $x + y = 2$, the hyperbola $x^2 - y^2 = 1$, and the hyperbola $y^2 - x^2 = 1$.
Hint: Use the change of variables from Problem 5.

b. The centroid of the triangle ΔPQR, where $P = (1, 0)$, $Q = (0, 1)$, $R = (2, 2)$.
Hint: Use a linear coordinate system, like the one in Problem 4.

In both parts, be very careful about the limits of integration!

Gottfried Wilhelm Leibniz
(1646–1716)



Johann I Bernoulli
(1667–1748)