Homework 5
Math 18400-Section 41, 51

Due: Wednesday February 16th

Exercise 1. Set up and evaluate triple integrals to compute the volumes of the following $3D$ regions.

1. The pyramid whose base is the square with vertices $(\pm 1, \pm 1, 0)$, and whose upper vertex is $(0, 0, 1)$.
2. The region cut out from a sphere of radius 2 by a cylinder of radius 1, in such a way that one diameter of the sphere lies on the surface of the cylinder.
   Hint: Center the sphere at the origin, orient the cylinder vertically, and use cylindrical coordinates.
3. The region inside the cone $z^2 = x^2 + y^2$ and between the horizontal planes $z = 1$ and $z = 2$.

For part **a**, check your answer using the standard formula for the volume of a pyramid:

$$V = \frac{1}{3} \cdot \text{Base} \cdot \text{Height}$$

For part **c**, check your answer by integrating in spherical coordinates (or in cylindrical coordinates, if you initially chose to integrate in spherical coordinates).

Exercise 2. Consider the three-dimensional region D which is defined by the following inequalities:

$$x^2 + z^2 \leq 1 \ , \ \ x^2 + y^2 \leq 1.$$

1. Let D_{x_0} be the 2D region obtained by slicing D with the plane $x = x_0$. For which values of x_0 is D_{x_0} nonempty? For every such value of x_0, show that D_{x_0} is a square, and identify its vertices.
2. Use part **a** to calculate the volume of D (use rectangular coordinates).
3. Sketch the $2D$ region R obtained by projecting D onto the xy plane. For (x_0, y_0) in R, identify the lower and upper limits of the line segment obtained by intersecting D with the line $x = x_0$, $y = y_0$.
4. Use part **c** to calculate the volume of D (use rectangular coordinates).
5. Set up the integral using the method of parts **c**/**d**, but in cylindrical coordinates. It is difficult but possible to evaluate the integral this way - anyone who succeeds gets extra credit!

Exercise 3. For each triple integral, sketch the region of integration and evaluate the integral.

1. $\iiint_A z e^{x^2+y^2} \, dV$, where A is the region defined by

$$1 \leq \sqrt{x^2 + y^2} \leq 2 \ , \ \ 0 \leq z \leq 1$$

2. $\iiint_B (x^2 + y^2 + z^2) dV$ where B is the region defined by

$$x^2 + y^2 \leq 1 \ , \ \ z^2 \leq 4(x^2 + y^2) \ , \ \ -2 \leq z \leq 2$$

3. $\iiint_C z dV$, where C is the region defined by

$$z \geq 0 \ , \ \ 1 \leq x^2 + y^2 + z^2 \leq 4$$

4. $\iiint_D (x^2+y^2+z^2)^{\frac{3}{2}} dV$, where D is the region defined by

$$x^2+y^2 \leq z^2\ ,\quad z \geq 0\ ,\quad x^2+y^2+z^2 \leq 1$$

Exercise 4. Consider the solid T with vertices $(-1,-1,-1)$, $(2,-1,-1)$, $(-1,2,-1)$, $(-1,-1,1)$, $(2,-1,1)$, and $(-1,2,1)$. T is a triangular prism, and its base is a right isosceles triangle on the plane $z=-1$.

1. Sketch T and determine its volume (without doing any integrals).

2. Integrate the functions x and y^2 over T.

3. Find the center of mass of T, assuming that it is made out of a material of unit density.

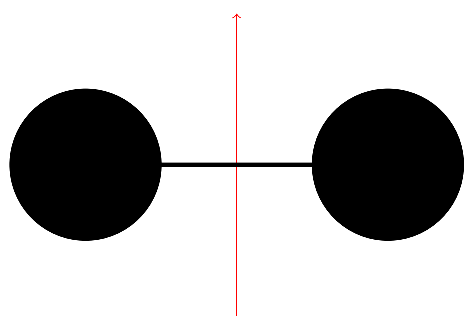
4. Find the moment of inertia of T around the z axis.

5. (Optional) Find the principal axes of T, and find the moment of inertia of T around each principal axis.

Hint: You don't need to do any extra integrals in parts ***c*** *and* ***d*** *- the remaining integrals you need to calculate can either be reduced to the ones in part* ***a*** *or evaluated intuitively, by applying principles of symmetry. For the definitions of center of mass and moment of inertia, see the notes for weeks 4 and 5.*

Exercise 5. **1.** Prove the parallel axis theorem: The moment of inertia I of any body about a given axis is $I = I_m + Md^2$, where M is the mass of the body, I_m is the moment of inertia of the body about an axis through the center of mass and parallel to the given axis, and d is the distance between the two axes.
Hint: Place the center of mass at the origin, and orient the object so that both axes are vertical.

2. A barbell is built out of two spheres of radius R and mass M, connected by a bar of length L:



Assuming that the mass of the bar is negligible, what is the moment of inertia of the barbell around the axis which passes through the centers of both spheres? What is the moment of inertia of the barbell around a perpendicular bisector of the bar (the axis shown in red)?
Assume the barbell has constant density.

Christiaan Huygens
(1629–1695)

Jakob Steiner
(1796–1863)