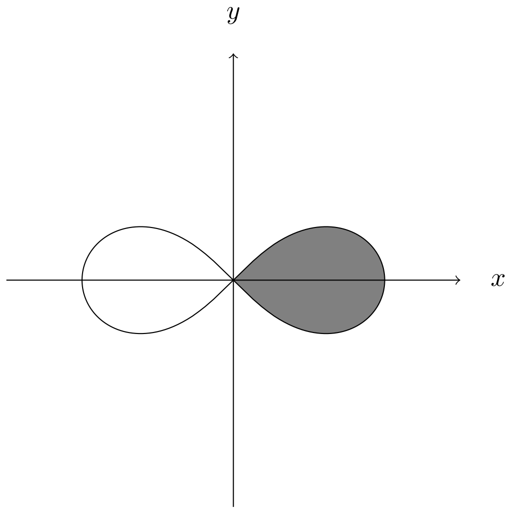
Homework 4
Math 18400-Section 41, 51

Due: Wednesday February 9th

Exercise 1. For each of the following domains, draw it and compute its area:

a. $-1 \leqslant x \leqslant 1$ and $x^2 \leqslant y \leqslant 4 - x^3$ **b.** $0 < x < \frac{\pi}{2}$ and $\sin(x) < y < x$
c. $0 < y < 1$ and $0 < x < e^y$ **d.** $0 < y < 1$ and $1 - \sqrt{y} < x < 1 + \sqrt{y}$
e. The area of D, where D is one of the two regions enclosed by the following curve: $(x^2 + y^2)^2 = x^2 - y^2$

Here is a picture of D:



Hint: Use polar coordinates, and be careful about the limits of integration.

Exercise 2. For each of the following questions, draw the domain of integration and integrate the given function over the domain.

1. $D : x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 1$ and $f(x, y) = xy(x + y)$.
2. $D : x \geqslant 0,\ y \geqslant 0,\ xy + x + y \leqslant 1$ and $f(x, y) = xy$.
3. $D : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ x^2 + y^2 \geqslant 1$ and $f(x, y) = \frac{xy}{1+x^2+y^2}$. *Hint: Prove that the antiderivative of* $u' \ln u$ *is* $u \ln u - u$.

Exercise 3. Consider the following iterated integral:

$$I = \int_0^1 \int_x^1 x^2 e^{-y^2} \, dy dx$$

In order to evaluate this integral explicitly, it is necessary to change the order of integration.

1. Draw a 2D region R such that $I = \iint_R x^2 e^{-y^2} \, dA$.
2. Express I as an iterated integral with the opposite order of integration ($dxdy$ instead of $dydx$).
3. Determine the value of I.

Exercise 4. The goal of this problem is to determine the value of the following integral:

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt.$$

We will do this using a clever trick which involves double integrals.

1. Let Q be the first quadrant in the xy plane (defined by the inequalities $x \geq 0$, $y \geq 0$). Show that

$$\iint_Q e^{-x^2-y^2}\,dA = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = K^2,$$

where

$$K = \int_0^{\infty} e^{-t^2}\,dt.$$

Hint: You don't need to evaluate integrals to do this, just use properties of exponents and integrals.

2. Evaluate the double integral in part **a**, by converting it to polar coordinates.

3. Show that

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}.$$

Jacques Bernoulli
(1654–1705)

Johann Carl Friedrich Gauss
(1777–1855)