Homework 3
Math 18400-Section 41, 51

Due: Wednesday February 2nd

Exercise 1. Find all critical points and their nature (local minimum, local maximum or saddle point) for each of the following functions:

1. $f(x,y) = y^2 - x^2 + \frac{x^4}{2}$

2. $f(x,y) = x^3 + y^3 - 3xy$

3. $f(x,y) = x^4 + y^4 - 4(x-y)^2$

4. $f(x,y) = y\left(x^2 + (\ln y)^2\right)$

Exercise 2. We consider the function $f(x,y) = y^2 - x^2y + x^2$ for x, y such that $x^2 - 1 \leqslant y \leqslant 1 - x^2$.

1. Draw the set of points D defined by $x^2 - 1 \leqslant y \leqslant 1 - x^2$.

2. Find the critical points of f.

3. Find the minimum and the maximum of f on the boundary of D.

4. Find the minimum and the maximum of f on D.

Exercise 3. Use Lagrange multipliers to solve the following constrained optimization problems:

1. Find the maximum possible volume of a cylinder which can be inscribed in a sphere of radius 1.

2. Find the maximum possible volume of a cylinder whose surface area is equal to 1.
Note: The surface area of the cylinder should include the top and bottom.

3. Find the maximum possible volume of a rectangular box which can be inscribed in a sphere of radius 1.

4. Find the maximum possible volume of a rectangular box whose surface area is equal to 1.

Exercise 4. Consider the function

$$f(x,y,z) = x^2 + 4xy + 3y^2 + 4yz + z^2$$

and let P be a point on the unit sphere ($x^2 + y^2 + z^2 = 1$) where $f(x,y,z)$ attains its maximum value.

1. Write this function in the form $f(\vec{x}) = \vec{x}^T A \vec{x}$, where A is a 3×3 symmetric matrix.

2. Show that gradient and Hessian matrix of f are given by $\vec{\nabla} f = 2A\vec{x}$ and $H_f = 2A$.

3. Does $f(x,y,z)$ have a local maximum, local minimum, or saddle point at the origin?

4. Show that the vector $\overrightarrow{OP}$ is an eigenvector for A. *Hint: Use Lagrange multipliers.*

5. Show that the eigenvalues of A are 1, -1, and 5.

6. What are the maximum and minimum values of $f(x,y,z)$ on the unit ball ($x^2 + y^2 + z^2 \leq 1$)?

Exercise 5. Consider the function $f(x,y) = \cos x \sin y$, on the square region D defined by the inequalities

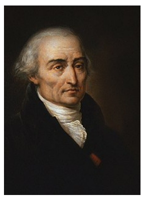
$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

1. Calculate the Hessian matrix of $f(x,y)$.

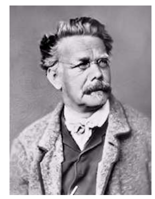
2. At which points (x,y) in D is the Hessian matrix degenerate?

3. At which points (x,y) in D is the Hessian matrix positive definite, negative definite, and indefinite? For each of these three possibilities, sketch the region where it occurs.

4. For values of (x,y) which are very close to $(\frac{\pi}{6}, \frac{\pi}{3})$, does the graph of $f(x,y)$ lie above its tangent plane, below its tangent plane, or neither?



Joseph Louis de Lagrange
(1768–1830)



Ludwig Otto Hesse
(1811–1874)