

Probabilité et Physique Statistique des Systèmes Désordonnés

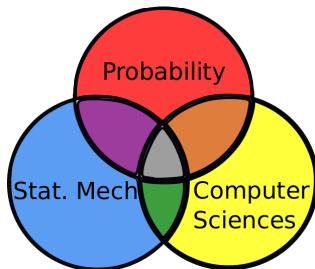
Louis-Pierre Arguin
Université de Montréal

Colloque CRM-ISM, Montréal
13 Janvier 2012



Outline

- 1 An Optimization Problem
- 2 The SK Model for Probabilists
- 3 An approximation
- 4 Recent Results



The Dean's Problem

Problem

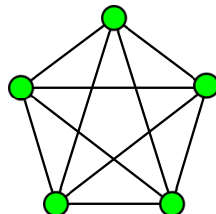
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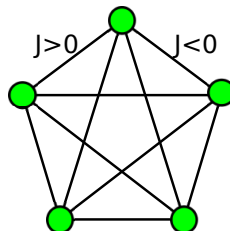


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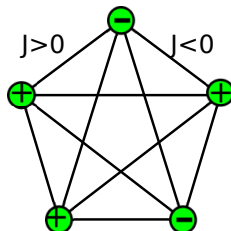


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 $\sigma(i)\sigma(j) = \text{sgn } J_{ij}$.



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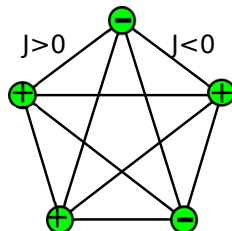
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- The pair (i, j) is satisfied iff
 $\sigma(i)\sigma(j) = \text{sgn } J_{ij}$.
- The functional to minimize is

$$H_{N,J}(\sigma) = - \sum_{i \neq j} J_{ij} \sigma(i)\sigma(j)$$

over configurations $\sigma \in \Sigma_N = \{-1, +1\}^N$:

$$\begin{aligned} \sigma : \{1, \dots, N\} &\rightarrow \{-1, +1\} \\ i &\mapsto \sigma(i) . \end{aligned}$$



The Dean's Problem

Question

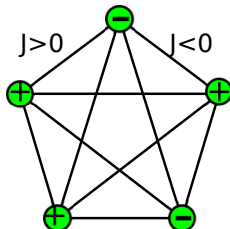
For N large,

- Describe minimum

$$\min_{\sigma \in \Sigma_N} H_{N,J}(\sigma) ?$$

- Describe minimizers

How different are two \sim optimal σ and σ' ?



- The Dean's problem is a **combinatorial optimization problem**.
Other examples: Traveling Salesman, K-SAT.
- Finding the minimizer of $H_{N,J}$ is **NP complete**.

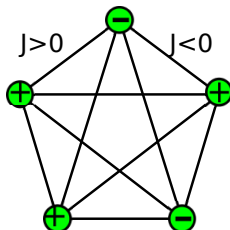
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For N large,

- Describe minimum $\min_{\sigma \in \Sigma_N} H_{N,J}(\sigma)$?
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How different are two \sim optimal σ and σ' ?

Why is it hard ?

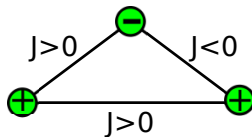


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Why is it hard ?

- Frustrations:
Odd number of $J_{ij} < 0$ in cycle \Rightarrow Some constraints are **unsatisfied**.
- A "Typical" J has some $J_{ij} < 0$.

Random Constraints and SK model

- \sim Typical:

$(J_{ij}; 1 \leq i < j \leq N)$ i.i.d. Gaussian r.v. mean 0 variance $\frac{1}{\sqrt{N}}$.

Disorder!

- This the Sherrington-Kirkpatrick model in statistical physics (1975):

$$H_{N,J}(\sigma) = - \sum_{i \neq j} J_{ij} \sigma(i) \sigma(j)$$

- It is a model of a spin glass on a complete graph.
Other graphs, say \mathbb{Z}^d , much harder !

The Probability Perspective

Write \mathbb{P} for the law of $J = (J_{ij})$ i.i.d. Gaussians.

$J \mapsto (H_{N,J}(\sigma), \sigma \in \Sigma_N)$ is a Gaussian process

- Covariance $\mathbb{E}[H_{N,J}(\sigma)H_{N,J}(\sigma')] = NR_N(\sigma, \sigma')^2 - 1$
- Overlap $R_N(\sigma, \sigma') = \frac{1}{N} \sum_{i=1}^N \sigma(i)\sigma'(i)$

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- **Overlap** $R_N(\sigma, \sigma') = \frac{1}{N} \sum_{i=1}^N \sigma(i)\sigma'(i)$
- The correlations are **regular**, **strong**, and **high-dimensional**.
- R_N is a symmetric, positive definite form on Σ_N (\sim inner product).
- R_N is related to the **Hamming distance** on Σ_N

$$R_N(\sigma, \sigma') = 1 - \frac{2\#\{i : \sigma(i) \neq \sigma'(i)\}}{N}$$

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- **Overlap** $R_N(\sigma, \sigma') = \frac{1}{N} \sum_{i=1}^N \sigma(i)\sigma'(i)$

Question

For N large,

- 1 *Describe the random variable* $\min_{\sigma \in \Sigma_N} H_{N,J}(\sigma)$?
- 2 *Describe the correlations* $R_N(\sigma, \sigma')$ for σ and $\sigma' \sim$ minimizers.

As probabilists, we are looking for **universal structures**.

The Statistical Physics Approach

Question

Does the limit $\frac{1}{N} \min_{\sigma \in \Sigma_N} H_{N,J}(\sigma)$ exist ?

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- **Gibbs Measure:** A probability measure on assignments Σ_N

$$G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)}, \quad \beta > 0$$

β is the inverse temperature.

Gibbs measure for β large: information on the minima of $(H_{N,J}(\sigma), \sigma \in \Sigma_N)$ and their relative location.

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- **Free Energy:** $f_{N,J}(\beta) = \frac{1}{N} \log \sum_{\sigma \in \Sigma_N} \exp -\beta H_{N,J}(\sigma)$.

Note that

$$-\frac{\log 2}{\beta} + \frac{\min H_{N,J}(\sigma)}{N} \leq \frac{-1}{\beta} f_{N,J}(\beta) \leq \frac{\min H_{N,J}(\sigma)}{N}$$

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GOAL: Understand the free energy and the Gibbs measure

Concentration of Measure Phenomenon

Lemma (Self-Averaging)

For J i.i.d. Gaussians of law \mathbb{P} ,

$$|f_{N,J}(\beta) - \mathbb{E}[f_{N,J}(\beta)]| \rightarrow 0 \text{ } \mathbb{P}\text{-almost surely as } N \rightarrow \infty$$

- Common instance: **the law of large number**.
- Here $f_{N,J}$ is not a linear function of J , but it is **Lipshitz**.
- Concentration of measure is a pillar of today's probability (see e.g. Ledoux-Talagrand).

An approximation

- Jensen's Inequality:

$$f_N(\beta) \leq \frac{1}{N} \log \mathbb{E} \left[\sum_{\sigma \in \Sigma_N} \exp -\beta H_{N,J}(\sigma) \right] = \log 2 + \frac{\beta^2}{2}$$

Equality should hold for β small.

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- Suppose that the variables $H_{N,J}(\sigma)$ are independent ! $R_N(\sigma, \sigma') = \delta_{\sigma\sigma'}$

$$\mathbb{P} \left(\min H_{N,J}(\sigma) > x \right) = \left(1 - \int_{-\infty}^x \frac{e^{-y^2/2N}}{\sqrt{2\pi N}} dy \right)^{2^N}$$

$$\min H_{N,J}(\sigma) = N\sqrt{2\log 2} + \text{Fluct.}$$

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Theorem (Derrida '81)

In the case where $(H_{N,J}(\sigma), \sigma \in \Sigma_N)$ are i.i.d. Gaussian variables $R_N(\sigma, \sigma') = \delta_{\sigma\sigma'}$

$$f(\beta) = \lim_{N \rightarrow \infty} \mathbb{E}[f_{N,J}(\beta)] = \begin{cases} \log 2 + \frac{\beta^2}{2} & \text{if } \beta \leq \sqrt{2 \log 2} \\ \beta \sqrt{2 \log 2} & \text{if } \beta > \sqrt{2 \log 2} . \end{cases}$$

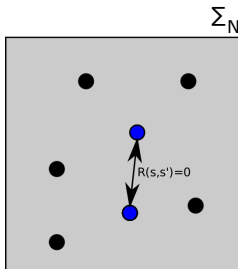
A Phase Transition

The approximation $R_N(\sigma, \sigma') = \delta_{\sigma\sigma'}$ is surprisingly complex:

- 1 **Phase transition:** $f(\beta)$ is singular at $\beta_c = \sqrt{2 \log 2}$
- 2 **Nature of the transition:**

$$\frac{d}{d\beta} f(\beta) \stackrel{\text{Convexity}}{=} \lim_{N \rightarrow \infty} \frac{d}{d\beta} \mathbb{E} f_{N,J}(\beta)$$

$$\stackrel{\text{Gaussian}}{=} \lim_{N \rightarrow \infty} \beta \mathbb{E} G_{\beta,N,J}^{\times 2} [1 - R_N(\sigma, \sigma')]$$



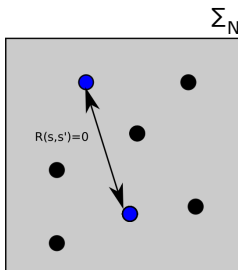
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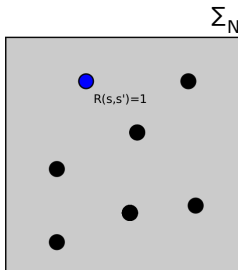
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The approximation $R_N(\sigma, \sigma') = \delta_{\sigma\sigma'}$ is surprisingly complex:

- ① **Phase transition:** $f(\beta)$ is singular at $\beta_c = \sqrt{2 \log 2}$
- ② **Transition of what ?:**

$$\begin{aligned} \frac{d}{d\beta} f(\beta) &\stackrel{\text{Convexity}}{=} \lim_{N \rightarrow \infty} \frac{d}{d\beta} \mathbb{E} f_{N,J}(\beta) \\ &\stackrel{\text{Gaussian}}{=} \lim_{N \rightarrow \infty} \beta \mathbb{E} G_{\beta,N,J}^{\times 2} [1 - R_N(\sigma, \sigma')] \end{aligned}$$

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E} G_{\beta,N,J}^{\times 2} [R_N(\sigma, \sigma')] &= \lim_{N \rightarrow \infty} \mathbb{E} G_{\beta,N,J}^{\times 2} \{\sigma = \sigma'\} \\ &= \begin{cases} 0 & \text{if } \beta \leq \sqrt{2 \log 2} \\ 1 - \frac{\sqrt{2 \log 2}}{\beta} & \text{if } \beta > \sqrt{2 \log 2} . \end{cases} \end{aligned}$$

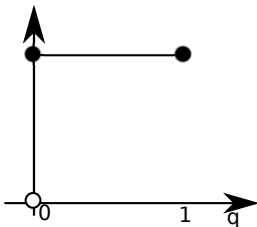
Some σ 's have macroscopic weight ! More than one.

A Phase Transition

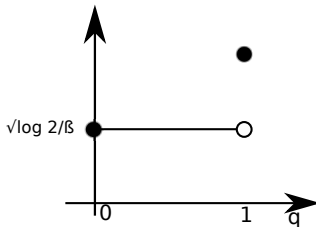
Order Parameter is a distribution function c.d.f.:

$$x_\beta(q) = \lim_{N \rightarrow \infty} \mathbb{E} G_{\beta, N, J}^{\times 2} \{R_N(\sigma, \sigma') \leq q\}$$

$\beta \leq \beta_c$



$\beta \geq \beta_c$



The Gibbs Measure

Universal results for $G_{\beta,N,J}$?

$(H_{N,\beta,J}(\sigma), \sigma \in \Sigma_N)$ Gaussian field with R_N generic.

Theorem (A & Chatterjee '10)

If $f(\beta)$ is differentiable at β , then the Gibbs measure is stochastically stable:

$$\left| \mathbb{E} \tilde{G}_{\beta,N,J}^{\times s} F(\sigma_1, \dots, \sigma_s) - \mathbb{E} G_{\beta,N,J}^{\times s} F(\sigma_1, \dots, \sigma_s) \right| \rightarrow 0$$

for

$$\tilde{G}_{\beta,N,J}(\sigma) = \frac{e^{g(\sigma)} \exp -\beta H_{N,\beta,J}(\sigma)}{\text{Norm.}}$$

g , a Gaussian field of covariance $R_N(\sigma, \sigma')$ independent of $H_{N,J}$.

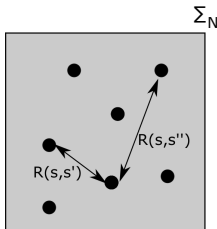
- Limit of $(G_{\beta,N,J})_N$ exists ? OK along subsequences

The Gibbs Measure

What do we learn from stochastic stability ?

Theorem (A & Chatterjee '10)

If the Gibbs measure is stochastically stable, then its limit measures are supported on an ∞ -dim. space (or on only one point).

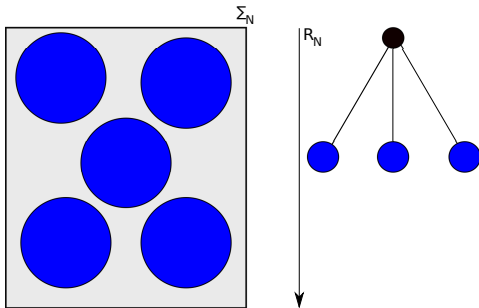


Theorem (Independent Case)

*If $R_N(\sigma, \sigma') = \delta_{\sigma, \sigma'}$, then $(G_{\beta, N, J})_N$ converges to an **atomic measure** with Poisson-Dirichlet weights.*

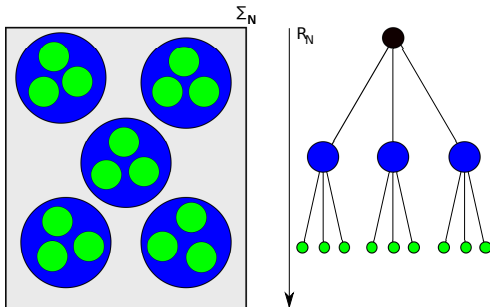
The Parisi Ultrametricity Conjecture

- The physicists (G. Parisi '80) proposed the following picture for the support of the Gibbs measure for high β :
- Solutions form equivalence classes of macroscopic weights.
- Distance is well defined and **ultrametric**.



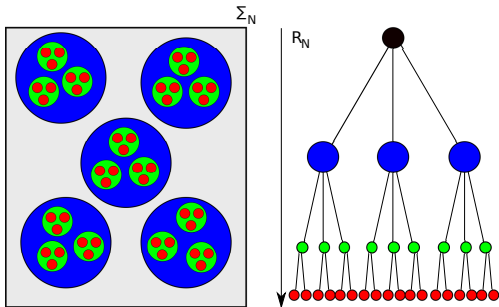
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- Distance is well defined and **ultrametric**.
- This structure repeats itself within classes.
- Again and again...

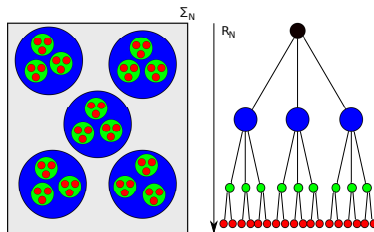


Optimal solutions are organized in a genealogical way. Why?

The Parisi Ultrametricity Conjecture

What is PROVED ?

- A & Aizenman '08:
 Stochastic stability (in a strong sense) \Rightarrow Parisi picture
 when $R_N(\sigma, \sigma')$ takes finite number of values (unif. in N).
- Panchenko ('09) SAME using strong moment identities.
- The whole structure is a Bolthausen-Sznitman coalescent.



- Strong: Holds for a perturbation of SK model but not SK.
- Enough to prove a formula for the free energy.

The Parisi Formula

Given $\psi \in C^2(\mathbb{R})$ and $x : [0, 1] \rightarrow [0, 1]$ a c.d.f.

Define the functional P_ψ by the evolution:

- $f(1, y) := \psi(y)$.
- $\partial_q f(q, y) + \frac{1}{2} (\partial_y^2 f(q, y) + x(q)(\partial_y f(q, y))^2) = 0$
- $P_\psi(x) = f(0, 0)$.

Theorem (Parisi Formula for the SK model)

$$\lim_{N \rightarrow \infty} f_N(\beta) = \min_{x: [0,1] \rightarrow [0,1] \text{ c.d.f.}} \{P_{\psi_1}(x) - P_{\psi_2}(x)\}$$

for $\psi_1(y) = \log 2 \cosh(\beta y)$ and $\psi_2(y) = \beta y$.

Order parameter

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- **Phase transition** at $\beta_c = 1$ (Toninelli '02)
- Below β_c same as IID (Aizenman Lebowitz Ruelle '87)
- Proved by Guerra and Talagrand '03 (**Not constructive**).
- **Constructive proof IF Parisi picture holds**: Panchenko '10, A&Chatterjee '10.

Open Questions

What is NOT PROVED ?

- What is $x(q)$ above $\beta > \beta_c$? **Conjecture: It is a continuous function!**
- Does the Parisi picture hold for pure SK (ok for perturbed model) ?
- What about SK model on a different graph, say \mathbb{Z}^d ?
Is the picture as complex !?!

